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Bethe vectors of quantum integrable models with $GL(3)$ trigonometric R-matrix

S. Belliard[a], S. Pakuliak[b], E. Ragoucy[c], N. A. Slavnov[d*]

[a] *Université Montpellier 2, Laboratoire Charles Coulomb, UMR 5221, F-34095 Montpellier, France*

[b] *Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow reg., Russia, Moscow Institute of Physics and Technology, 141700, Dolgoprudny, Moscow reg., Russia, Institute of Theoretical and Experimental Physics, 117259 Moscow, Russia*

[c] *Laboratoire de Physique Théorique LAPTH, CNRS and Université de Savoie, BP 110, 74941 Annecy-le-Vieux Cedex, France*

[d] *Steklov Mathematical Institute, Moscow, Russia*

Abstract

We study quantum integrable models with $GL(3)$ trigonometric R-matrix solvable by the nested algebraic Bethe ansatz. Using presentation of the universal Bethe vectors in terms of projections of the product of the currents of quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_3)$ onto intersections of different type Borel subalgebras, we prove that the set of the nested Bethe vectors is closed under action of the matrix elements of monodromy.

1 Introduction

We consider a quantum integrable model defined by the monodromy matrix $T(u)$ with matrix elements $T_{ij}(u)$, $i,j = 1,2,3$ which satisfies the commutation relation

$$\mathrm{R}(u,v)\cdot(T(u)\otimes\mathbf{1})\cdot(\mathbf{1}\otimes T(v)) = (\mathbf{1}\otimes T(v))\cdot(T(u)\otimes\mathbf{1})\cdot\mathrm{R}(u,v), \tag{1.1}$$

*samuel.belliard@univ-montp2.fr, pakuliak@theor.jinr.ru, eric.ragoucy@lapth.cnrs.fr, nslavnov@mi.ras.ru

with $GL(3)$ trigonometric quantum R-matrix

$$\begin{aligned}\mathrm{R}(u,v) \;=\; \mathsf{f}(u,v) \sum_{1\le i\le 3} \mathsf{E}_{ii}\otimes\mathsf{E}_{ii} \;+\; \sum_{1\le i<j\le 3} (\mathsf{E}_{ii}\otimes\mathsf{E}_{jj}+\mathsf{E}_{jj}\otimes\mathsf{E}_{ii}) \\ +\; \sum_{1\le i<j\le 3} (u\mathsf{g}(u,v)\mathsf{E}_{ij}\otimes\mathsf{E}_{ji}+v\mathsf{g}(u,v)\mathsf{E}_{ji}\otimes\mathsf{E}_{ij})\,.\end{aligned} \tag{1.2}$$

Here the rational functions $\mathsf{f}(u,v)$ and $\mathsf{g}(u,v)$ are

$$\mathsf{f}(u,v)=\frac{qu-q^{-1}v}{u-v},\quad \mathsf{g}(u,v)=\frac{(q-q^{-1})}{u-v}\,. \tag{1.3}$$

and $(\mathsf{E}_{ij})_{lk}=\delta_{il}\delta_{jk}$, $i,j,l,k=1,2,3$ are 3×3 matrices with unit in the intersection of ith row and jth column and zero matrix elements elsewhere. The R-matrix (1.2) is called "trigonometric" because its classical limit gives the classical trigonometric r-matrix [1]. The trigonometric R-matrix (1.2) is written in multiplicative variables and depends actually on the ratio u/v of these multiplicative parameters.

Due to the commutation relation (1.1) the transfer matrix $t(u)=\mathrm{tr}\,T(u)=T_{11}(u)+T_{22}(u)+T_{33}(u)$ generates a set of commuting integrals of motion and the first step of the algebraic Bethe ansatz [2] is the construction of the set of eigenstates for these commuting operators in terms of the monodromy matrix entries. We assume that these matrix elements act in a quantum space V and this space possesses a vector $|0\rangle\in V$ such that

$$T_{ij}(u)|0\rangle=0,\quad i>j,\quad T_{ii}(u)|0\rangle=\lambda_i(u)|0\rangle\,,\qquad \lambda_i(u)\in\mathbb{C}[[u,u^{-1}]]\,. \tag{1.4}$$

The eigenstates $\mathbb{B}^{a,b}(\bar u;\bar v)$ of the transfer matrix $t(u)$ in quantum integrable models with $GL(3)$ trigonometric R-matrix depend on two sets of the variables

$$\bar u=\{u_1,\dots,u_a\}\,,\quad \bar v=\{v_1,\dots,v_b\}\,, \tag{1.5}$$

which are called the Bethe parameters. These eigenstates can be constructed in the framework of the nested Bethe ansatz method formulated in [3] and are given by certain polynomials in monodromy matrix elements $T_{12}(u)$, $T_{23}(u)$, $T_{13}(u)$ with rational coefficients depending on the Bethe parameters.

In pioneer papers on nested Bethe ansatz [3, 4, 5] no explicit formulas for the Bethe vectors were obtained. The method in its original formulation allows one to get Bethe equations as conditions that Bethe vectors are eigenstates of the transfer matrix. Nevertheless, even when Bethe parameters are free and do not satisfy any restrictions, the structure of the Bethe vector (sometimes such Bethe vectors are called off-shell) is rather complicated. More explicit formulas for the off-shell nested Bethe vectors were obtained in [6] in the theory of solutions of the quantum Knizhnik–Zamolodchikov equation. The Bethe vectors were given by certain trace over auxiliary spaces of the products of the monodromy matrices and R-matrices. This presentation allows one to investigate the structure of the nested off-shell Bethe vectors and to obtain the explicit formulas for the nested Bethe vectors when the space V becomes the tensor product of the evaluation representations of the Yangian and of the positive Borel subalgebra of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ [7].

Explicit expressions for the off-shell nested Bethe vectors in the $GL(N)$ quantum integrable models in terms of the monodromy matrix elements were obtained in the papers [8, 9, 10], where the realization of these vectors in terms of the current generators of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ [11] were used. This realization uses the notion of projections onto intersection of the different type Borel subalgebras in the quantum affine algebras introduced firstly in [12] and an isomorphism between current [13] and L-operator formulation of the quantum affine algebras [14] investigated in [15].

Quite analogously one can construct dual off-shell Bethe vectors $\mathbb{C}^{a,b}(\bar{u};\bar{v})$ defined in the dual space V^* with the dual vacuum vector $\langle 0| \in V^*$:

$$\langle 0|T_{ij}(u) = 0, \quad i < j, \quad \langle 0|T_{ii}(u) = \lambda_i(u)\langle 0|\,, \tag{1.6}$$

which can be also explicitly written as polynomials in the monodromy matrix elements $T_{21}(u)$, $T_{32}(u)$, $T_{31}(u)$ with rational coefficients using current realization of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_3)$ [16].

For the class of nested quantum integrable models where the inverse scattering problem can be solved and local operators can be expressed in terms of the monodromy matrix elements [17], one can now address the problem of calculation of the form factors and the correlation functions of local operators. It was done in [18] for the quantum integrable models with $GL(2)$ trigonometric R-matrix, using determinant formulas for the scalar products of the Bethe vectors obtained in [19].

To approach this problem one has to answer the following question. Whether the action of the monodromy matrix elements onto nested off-shell Bethe vectors produces linear combinations of vectors with the same structure. If this is true, then the problem of computing the form factors of local operators can be reduced to the calculation of the scalar product between off-shell and on-shell[1] Bethe vectors. Moreover, since right and left Bethe vectors are presented as linear combinations of products of monodromy matrix elements, the calculation of these scalar products itself can be also reduced to the application of the action formulas of the monodromy matrix elements onto Bethe vectors.

The goal of this paper is to give a positive answer to this question and to present and prove the explicit formulas for such an action. We should say that in case of quantum integrable models with $GL(2)$ R-matrix, the question about action formulas is almost trivial, since the right and left off-shell Bethe vectors in this case are given by the product of the monodromy matrix elements $T_{12}(u)$ and $T_{21}(u)$ respectively. These action formulas can be easily extracted from the RTT relation (1.1) for the monodromy operators. In higher-rank systems, due to the nontrivial structure of nested Bethe vectors, application of the RTT relations for the calculation of the action formulas becomes a very complicated combinatorial problem. To solve it we will use the presentation of nested off-shell Bethe vectors in terms of the current generators of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_3)$ (2.7) and the relation between monodromy matrix elements and the current generators given by the Gauss decomposition (3.4).

[1]These are the Bethe vectors with parameters satisfying the Bethe equations.

2 Main results

2.1 Notations

To save space and simplify the presentation of formulas, we use following convention for the products of the commuting entries of the monodromy matrix $T_{ij}(w)$, vacuum eigenvalues $\lambda_i(w)$ and their ratios $\mathsf{r}_k(w) = \lambda_k(w)/\lambda_2(w)$, $k = 1, 3$. Namely, whenever such an operator or scalar function depends on a set of variables (for instance, $T_{ij}(\bar w)$, $\lambda_i(\bar u)$, $\mathsf{r}_k(\bar v)$), this means that we deal with the product of the operators or scalar functions with respect to the corresponding set:

$$T_{ij}(\bar w) = \prod_{w_k\in\bar w} T_{ij}(w_k); \quad \lambda_2(\bar u) = \prod_{u_j\in\bar u} \lambda_2(u_j); \quad \mathsf{r}_k(\bar v_\ell) = \prod_{\substack{v_j\in\bar v\\ v_j\neq v_\ell}} \mathsf{r}_k(v_j). \tag{2.1}$$

Similar convention will be used for the products of functions $\mathsf{f}(u, v)$ and $\mathsf{g}(u, v)$

$$\mathsf{f}(w_i, \bar w_i) = \prod_{\substack{w_j\in\bar w\\ w_j\neq w_i}} \mathsf{f}(w_i, w_j); \quad \mathsf{g}(\bar u, \bar v) = \prod_{u_j\in\bar u} \prod_{v_k\in\bar v} \mathsf{g}(u_j, v_k), \tag{2.2}$$

The notation $\bar v_\ell$ for arbitrary set $\bar v$ means the set $\bar v \setminus \{v_\ell\}$. We will also use the sets $\bar w_{<j} = \{w_1, ..., w_{j-1}\}$ and $\bar w_{>j} = \bar w_j \setminus \bar w_{<j}$ with obvious convention for the products. Partitions of sets will be noted as $\bar u \Rightarrow \{\bar u_{\rm I}\,, \bar u_{\rm II}\}$.

To simplify further formulas we will introduce a special notation for product of non-commuting currents. Postponing the exact definition to section 3 (see definition (3.8) and commutation relations (3.9)–(3.17)), we will use the notation:

$$\mathcal{F}_1(\bar u) = F_1(u_a)F_1(u_{a-1})\cdots F_1(u_1), \qquad \mathcal{F}_2(\bar v) = F_2(v_b)\cdots F_2(v_2)F_2(v_1) \tag{2.3}$$

and

$$\begin{aligned} \mathcal{F}_1(\bar u_j) &= F_1(u_a)\cdots F_1(u_{j+1})F_1(u_{j-1})\cdots F_1(u_1),\\ \mathcal{F}_2(\bar v_i) &= F_2(v_b)\cdots F_2(v_{i+1})F_2(v_{i-1})\cdots F_2(v_1)\,. \end{aligned} \tag{2.4}$$

These notations are in accordance with the one used for commuting objects, except that now one needs to specify the order as prescribed in (2.3) and (2.4).

In various formulas below the Izergin determinant $\mathsf{K}_k(\bar x|\bar y)$ appears [20]. It is defined for two sets $\bar x$ and $\bar y$ of the same cardinality $\#\bar x = \#\bar y = k$:

$$\mathsf{K}_k(\bar x|\bar y) = \frac{\prod_{1\le i,j\le k}(qx_i - q^{-1}y_j)}{\prod_{1\le i<j\le k}(x_i - x_j)(y_j - y_i)} \cdot \det\left[\frac{q - q^{-1}}{(x_i - y_j)(qx_i - q^{-1}y_j)}\right]. \tag{2.5}$$

Below we also use two modifications of the Izergin determinant

$$\mathsf{K}^{(l)}_k(\bar x|\bar y) = \prod_{i=1}^{k} x_i \cdot \mathsf{K}_k(\bar x|\bar y)\,, \qquad \mathsf{K}^{(r)}_k(\bar x|\bar y) = \prod_{i=1}^{k} y_i \cdot \mathsf{K}_k(\bar x|\bar y)\,. \tag{2.6}$$

Some properties of the Izergin determinant and its modifications are gathered into Appendix A.

2.2 Explicit expression for Bethe vectors

The right and left off-shell Bethe vectors can be presented using current realization of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_3)$ given in Section 3

$$\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \frac{\beta(\bar{u}|\bar{v})}{\mathsf{f}(\bar{v},\bar{u})} P_f^+ \left(F_2(v_b)\cdots F_2(v_1)\cdot F_1(u_a)\cdots F_1(u_1)\right)\cdot \mathsf{r}_3(\bar{v})\;|0\rangle\,, \tag{2.7}$$

$$\mathbb{C}^{a,b}(\bar{u},\bar{v}) = \frac{\beta(\bar{u}|\bar{v})}{\mathsf{f}(\bar{v},\bar{u})}\;\langle 0|\;\mathsf{r}_3(\bar{v}) P_e^+ \left(E_1(u_1)\cdots E_1(u_a)\cdot E_2(v_1)\cdots E_2(v_b)\right)\,, \tag{2.8}$$

where

$$\beta(\bar{u}|\bar{v}) = \prod_{1\le\ell<\ell'\le a} \mathsf{f}(u_{\ell'},u_\ell) \prod_{1\le\ell<\ell'\le b} \mathsf{f}(v_{\ell'},v_\ell)\,,$$

and P_f^+ and P_e^+ are projection onto subalgebras of $U_q(\widehat{\mathfrak{gl}}_3)$ generated by the non-negative and positive modes of the simple root currents $F_i(u)$ and $E_i(u)$, $i=1,2$, respectively. These operators of projection onto subalgebras in the positive Borel subalgebra of $U_q(\widehat{\mathfrak{gl}}_3)$ were introduced in [12] and their detailed theory was developed in [11]. The formal definition of these projections is given in the present paper by the formulas (3.24) and (3.25).

Note that the function $\beta(\bar{u}|\bar{v})$ removes all poles and zeros which originate from the product of the same type currents, while the product of the functions $\mathsf{f}(\bar{v},\bar{u})$ removes all poles which originate from the product of the different type currents. The product $F_i(u_2)F_i(u_1)$ has simple pole at the point $u_1=q^2u_2$ and simple zero at $u_1=u_2$ and the product $F_2(v)F_1(u)$ has simple pole at the point $u=v$. These 'analytical' properties of the product of the currents are defined by the commutation relations (3.13), (3.14) and were explained in details in the papers [8, 9] using the notion of the ordering of the current generators.

2.3 Multiple action of $T_{ij}(\bar{w})$ operators on Bethe vectors

Now we give a list of multiple actions of the operators $T_{ij}(\bar{w})$ onto Bethe vector $\mathbb{B}^{a,b}(\bar{u};\bar{v})$. Below everywhere $\{\bar{v},\bar{w}\}=\bar{\xi}$, $\{\bar{u},\bar{w}\}=\bar{\eta}$ and $\#\bar{w}=n$.

- Multiple action of T_{13}

$$T_{13}(\bar{w})\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{w})\,\mathbb{B}^{a+n,b+n}(\bar{\eta};\bar{\xi}). \tag{2.9}$$

- Multiple action of T_{12}

$$T_{12}(\bar{w})\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{w}) \sum \frac{\mathsf{f}(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I})}{\mathsf{f}(\bar{w},\bar{\xi}_{\rm I})} \mathsf{K}_n^{(r)}(\bar{w}|\bar{\xi}_{\rm I})\,\mathbb{B}^{a+n,b}(\bar{\eta};\bar{\xi}_{\rm II}). \tag{2.10}$$

The sum is taken over partitions of $\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}$ with $\#\bar{\xi}_{\rm I}=n$.

- Multiple action of T_{23}

$$T_{23}(\bar{w})\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{w}) \sum \frac{\mathsf{f}(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II})}{\mathsf{f}(\bar{\eta}_{\rm I},\bar{w})} \mathsf{K}_n^{(l)}(\bar{\eta}_{\rm I}|\bar{w})\,\mathbb{B}^{a,b+n}(\bar{\eta}_{\rm II};\bar{\xi}). \tag{2.11}$$

The sum is taken over partitions of $\bar{\eta}\Rightarrow\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm II}\}$ with $\#\bar{\eta}_{\rm I}=n$.

- Multiple action of T_{22}

$$T_{22}(\bar w)\mathbb{B}^{a,b}(\bar u;\bar v)=\lambda_2(\bar w)\sum\frac{\mathsf{f}(\bar\xi_{\rm II},\bar\xi_{\rm I})\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm II})}{\mathsf{f}(\bar w,\bar\xi_{\rm I})\mathsf{f}(\bar\eta_{\rm I},\bar w)}\mathsf{K}_n^{(r)}(\bar w|\bar\xi_{\rm I})\,\mathsf{K}_n^{(l)}(\bar\eta_{\rm I}|\bar w)\,\mathbb{B}^{a,b}(\bar\eta_{\rm II};\bar\xi_{\rm II}). \tag{2.12}$$

The sum is taken over partitions of: $\bar\eta\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm II}\}$ with $\#\bar\eta_{\rm I}=n$;
$\bar\xi\Rightarrow\{\bar\xi_{\rm I},\bar\xi_{\rm II}\}$ with $\#\bar\xi_{\rm I}=n$.

- Multiple action of T_{11}

$$T_{11}(\bar w)\mathbb{B}^{a,b}(\bar u;\bar v)=\lambda_2(\bar w)\sum\frac{\mathsf{r}_1(\bar\eta_{\rm I})}{\mathsf{f}(\bar\xi_{\rm II},\bar\eta_{\rm I})}\frac{\mathsf{f}(\bar\xi_{\rm II},\bar\xi_{\rm I})\mathsf{f}(\bar\eta_{\rm II},\bar\eta_{\rm I})}{\mathsf{f}(\bar w,\bar\xi_{\rm I})\mathsf{f}(\bar\xi_{\rm I},\bar\eta_{\rm I})}\mathsf{K}_n^{(r)}(\bar w|\bar\xi_{\rm I})\,\mathsf{K}_n^{(r)}(\bar\xi_{\rm I}|\bar\eta_{\rm I})\,\mathbb{B}^{a,b}(\bar\eta_{\rm II};\bar\xi_{\rm II}). \tag{2.13}$$

The sum is taken over partitions of: $\bar\eta\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm II}\}$ with $\#\bar\eta_{\rm I}=n$;
$\bar\xi\Rightarrow\{\bar\xi_{\rm I},\bar\xi_{\rm II}\}$ with $\#\bar\xi_{\rm I}=n$.

- Multiple action of T_{33}

$$T_{33}(\bar w)\mathbb{B}^{a,b}(\bar u;\bar v)=\lambda_2(\bar w)\sum\frac{\mathsf{r}_3(\bar\xi_{\rm I})}{\mathsf{f}(\bar\xi_{\rm I},\bar\eta_{\rm II})}\frac{\mathsf{f}(\bar\xi_{\rm I},\bar\xi_{\rm II})\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm II})}{\mathsf{f}(\bar\xi_{\rm I},\bar\eta_{\rm I})\mathsf{f}(\bar\eta_{\rm I},\bar w)}\mathsf{K}_n^{(l)}(\bar\eta_{\rm I}|\bar w)\,\mathsf{K}_n^{(l)}(\bar\xi_{\rm I}|\bar\eta_{\rm I})\,\mathbb{B}^{a,b}(\bar\eta_{\rm II};\bar\xi_{\rm II}). \tag{2.14}$$

The sum is taken over partitions of: $\bar\eta\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm II}\}$ with $\#\bar\eta_{\rm I}=n$;
$\bar\xi\Rightarrow\{\bar\xi_{\rm I},\bar\xi_{\rm II}\}$ with $\#\bar\xi_{\rm I}=n$.

- Multiple action of T_{21}

$$\begin{aligned}T_{21}(\bar w)\mathbb{B}^{a,b}(\bar u;\bar v)=\lambda_2(\bar w)\sum\mathsf{r}_1(\bar\eta_{\rm I})\,&\frac{\mathsf{f}(\bar\eta_{\rm II},\bar\eta_{\rm I})\mathsf{f}(\bar\eta_{\rm II},\bar\eta_{\rm III})\mathsf{f}(\bar\eta_{\rm III},\bar\eta_{\rm I})\mathsf{f}(\bar\xi_{\rm II},\bar\xi_{\rm I})}{\mathsf{f}(\bar\xi,\bar\eta_{\rm I})\mathsf{f}(\bar w,\bar\xi_{\rm I})\mathsf{f}(\bar\eta_{\rm II},\bar w)}\\&\times\mathsf{K}_n^{(l)}(\bar\eta_{\rm II}|\bar w)\,\mathsf{K}_n^{(r)}(\bar\xi_{\rm I}|\bar\eta_{\rm I})\,\mathsf{K}_n^{(r)}(\bar w|\bar\xi_{\rm I})\;\mathbb{B}^{a-n,b}(\bar\eta_{\rm III};\bar\xi_{\rm II}).\end{aligned} \tag{2.15}$$

The sum is taken over partitions of: $\bar\eta\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm II},\bar\eta_{\rm III}\}$ with $\#\bar\eta_{\rm I}=\#\bar\eta_{\rm II}=n$;
$\bar\xi\Rightarrow\{\bar\xi_{\rm I},\bar\xi_{\rm II}\}$ with $\#\bar\xi_{\rm I}=n$.

- Multiple action of T_{32}

$$\begin{aligned}T_{32}(\bar w)\mathbb{B}^{a,b}(\bar u;\bar v)=\lambda_2(\bar w)\sum\mathsf{r}_3(\bar\xi_{\rm I})\,&\frac{\mathsf{f}(\bar\xi_{\rm I},\bar\xi_{\rm II})\mathsf{f}(\bar\xi_{\rm I},\bar\xi_{\rm III})\mathsf{f}(\bar\xi_{\rm III},\bar\xi_{\rm II})\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm II})}{\mathsf{f}(\bar\xi_{\rm I},\bar\eta)\mathsf{f}(\bar\eta_{\rm I},\bar w)\mathsf{f}(\bar w,\bar\xi_{\rm II})}\\&\times\mathsf{K}_n^{(l)}(\bar\eta_{\rm I}|\bar w)\,\mathsf{K}_n^{(l)}(\bar\xi_{\rm I}|\bar\eta_{\rm I})\,\mathsf{K}_n^{(r)}(\bar w|\bar\xi_{\rm II})\;\mathbb{B}^{a,b-n}(\bar\eta_{\rm II};\bar\xi_{\rm III}).\end{aligned} \tag{2.16}$$

The sum is taken over partitions of: $\bar\xi\Rightarrow\{\bar\xi_{\rm I},\bar\xi_{\rm II},\bar\xi_{\rm III}\}$ with $\#\bar\xi_{\rm I}=\#\bar\xi_{\rm II}=n$;
$\bar\eta\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm II}\}$ with $\#\bar\eta_{\rm I}=n$.

- Multiple action of T_{31}

$$\begin{aligned}T_{31}(\bar w)\mathbb{B}^{a,b}(\bar u;\bar v)=\lambda_2(\bar w)\sum&\,\mathsf{r}_1(\bar\eta_{\rm II})\,\mathsf{r}_3(\bar\xi_{\rm I})\,\mathsf{K}_n^{(l)}(\bar\xi_{\rm I}|\bar\eta_{\rm I})\,\mathsf{K}_n^{(r)}(\bar\xi_{\rm II}|\bar\eta_{\rm II})\,\mathsf{K}_n^{(l)}(\bar\eta_{\rm I}|\bar w)\,\mathsf{K}_n^{(r)}(\bar w|\bar\xi_{\rm II})\\&\times\frac{\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm II})\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm III})\mathsf{f}(\bar\eta_{\rm III},\bar\eta_{\rm II})\mathsf{f}(\bar\xi_{\rm I},\bar\xi_{\rm II})\mathsf{f}(\bar\xi_{\rm I},\bar\xi_{\rm III})\mathsf{f}(\bar\xi_{\rm III},\bar\xi_{\rm II})}{\mathsf{f}(\bar\xi_{\rm I},\bar\eta)\mathsf{f}(\bar\xi_{\rm III},\bar\eta_{\rm II})\mathsf{f}(\bar\xi_{\rm II},\bar\eta_{\rm II})\mathsf{f}(\bar\eta_{\rm I},\bar w)\mathsf{f}(\bar w,\bar\xi_{\rm II})}\,\mathbb{B}^{a-n,b-n}(\bar\eta_{\rm III};\bar\xi_{\rm III}).\end{aligned} \tag{2.17}$$

Note that the product of the rational functions $\mathsf{f}(\bar\xi_{\rm I},\bar\eta)\mathsf{f}(\bar\xi_{\rm III},\bar\eta_{\rm II})\mathsf{f}(\bar\xi_{\rm II},\bar\eta_{\rm II})$ in the denominator of the r.h.s. of (2.17) can be equally rewritten as $\mathsf{f}(\bar\xi,\bar\eta_{\rm II})\mathsf{f}(\bar\xi_{\rm I},\bar\eta_{\rm I})\mathsf{f}(\bar\xi_{\rm I},\bar\eta_{\rm III})$.

The sum is taken over partitions of: $\bar\xi\Rightarrow\{\bar\xi_{\rm I},\bar\xi_{\rm II},\bar\xi_{\rm III}\}$ with $\#\bar\xi_{\rm I}=\#\bar\xi_{\rm II}=n$;
$\bar\eta\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm II},\bar\eta_{\rm III}\}$ with $\#\bar\eta_{\rm I}=\#\bar\eta_{\rm II}=n$.

Proof of the formulas (2.9)–(2.17) will be divided into two steps. First, we will prove these formulas using the current approach and presentation of the off-shell Bethe vectors in the form (2.7) in case of the action of one monodromy element, that is $\#\bar w=n=1$. Then we will use the induction to prove these formulas for $n>1$.

In this paper we will use the current techniques to prove the action formulas. For that we will use a special presentation of the projection of the product of the full currents in (2.7) found in the paper [21], the expression of the monodromy elements in term of the Gauss coordinates given by the formulas (3.4), the commutation relations of the Gauss coordinates with the currents and several properties of the projections.

3 Quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_3)$

3.1 Two realizations of $U_q(\widehat{\mathfrak{gl}}_3)$

Quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_3)$ is an associative algebra with unit. In the L-operator formulation [14] it is generated by the modes $L^{\pm}_{ij}[n]$, $i,j=1,2,3$, $n\ge 0$ such that

$$L^{+}_{ji}[0]=L^{-}_{ij}[0]=0,\qquad 1\le i<j\le 3\,. \tag{3.1}$$

These modes can be gathered into generation series[2]

$$L^{\pm}(u)=\sum_{n\ge 0}\sum_{i,j=1}^{3}\mathsf{E}_{ij}\otimes L^{\pm}_{ij}[n]u^{\mp n}\in {\rm End}(\mathbb{C}^3)\otimes U_q(\mathfrak{b}_\pm)\,, \tag{3.2}$$

where $U_q(\mathfrak{b}_\pm)\subset U_q(\widehat{\mathfrak{gl}}_3)$ are the positive and negative Borel subalgebras of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_3)$. These generating series can be called universal monodromy matrices since they satisfy the same as (1.1) commutation relation

$$\mathrm{R}(u,v)\cdot(L^{\mu}(u)\otimes\mathbf{1})\cdot(\mathbf{1}\otimes L^{\nu}(v))=(\mathbf{1}\otimes L^{\nu}(v))\cdot(L^{\mu}(u)\otimes\mathbf{1})\cdot\mathrm{R}(u,v), \tag{3.3}$$

where $\mu,\nu=\pm$.

Quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_3)$ is a Hopf algebra and the Borel subalgebras generated by the modes of the L-operators $L^{\pm}(u)$ are the Hopf subalgebras with the standard coproduct

$$\Delta\left(L^{\pm}_{ij}(u)\right)=\sum_{k=1}^{3}L^{\pm}_{kj}(u)\otimes L^{\pm}_{ik}(u)\,.$$

[2]There is also one relation for the zero modes of the diagonal matrix elements of L-operators $L^{+}_{jj}[0]L^{-}_{jj}[0]=1$, $j=1,2,3$, which is not important for our considerations.

In what follows we will need another realization of the same algebra, so-called current realization of quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_3)$ given in [13]. To relate current and L-operator realizations of the same algebra we introduce according to [15] the Gauss decomposition of the L-operator (3.2)

$$L^{\pm}(u)=\begin{pmatrix}1 & \mathrm{F}^{\pm}_{21}(u) & \mathrm{F}^{\pm}_{31}(u)\\ 0 & 1 & \mathrm{F}^{\pm}_{32}(u)\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}k^{\pm}_1(u) & 0 & 0\\ 0 & k^{\pm}_2(u) & 0\\ 0 & 0 & k^{\pm}_3(u)\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ \mathrm{E}^{\pm}_{12}(u) & 1 & 0\\ \mathrm{E}^{\pm}_{13}(u) & \mathrm{E}^{\pm}_{23}(u) & 1\end{pmatrix}, \tag{3.4}$$

or

$$L^{\pm}_{ab}(u)=\mathrm{F}^{\pm}_{ba}(u)k^{+}_{b}(u)+\sum_{b<m\le 3}\mathrm{F}^{\pm}_{ma}(u)k^{+}_{m}(t)\mathrm{E}^{\pm}_{bm}(u),\qquad a<b, \tag{3.5}$$

$$L^{\pm}_{bb}(u)=k^{\pm}_{b}(u)+\sum_{b<m\le 3}\mathrm{F}^{\pm}_{mb}(u)k^{\pm}_{m}(u)\mathrm{E}^{\pm}_{bm}(u), \tag{3.6}$$

$$L^{\pm}_{ab}(u)=k^{\pm}_{a}(u)\mathrm{E}^{\pm}_{ba}(u)+\sum_{a<m\le 3}\mathrm{F}^{\pm}_{ma}(u)k^{\pm}_{m}(u)\mathrm{E}^{\pm}_{bm}(u),\qquad a>b\,. \tag{3.7}$$

It was proved in the paper [15] that after substitution of the decompositions (3.5)–(3.7) into the commutation relations (3.3) one can obtain for the linear combinations of the Gauss coordinates

$$F_i(t)=\mathrm{F}^{+}_{i+1\,i}(t)-\mathrm{F}^{-}_{i+1\,i}(t)\,,\quad E_i(t)=\mathrm{E}^{+}_{i\,i+1}(t)-\mathrm{E}^{-}_{i\,i+1}(t) \tag{3.8}$$

and $k^{\pm}_i(t)$ the following commutation relations:

$$(q^{-1}z-qw)E_i(z)E_i(w)=E_i(w)E_i(z)(qz-q^{-1}w)\,, \tag{3.9}$$

$$(z-w)E_i(z)E_{i+1}(w)=E_{i+1}(w)E_i(z)(q^{-1}z-qw)\,, \tag{3.10}$$

$$k^{\pm}_i(z)E_i(w)\left(k^{\pm}_i(z)\right)^{-1}=\frac{z-w}{q^{-1}z-qw}E_i(w)\,, \tag{3.11}$$

$$k^{\pm}_{i+1}(z)E_i(w)\left(k^{\pm}_{i+1}(z)\right)^{-1}=\frac{z-w}{qz-q^{-1}w}E_i(w)\,, \tag{3.12}$$

$$k^{\pm}_i(z)E_j(w)\left(k^{\pm}_i(z)\right)^{-1}=E_j(w),\qquad \text{if}\quad i\ne j,j+1\,,$$

$$(qz-q^{-1}w)F_i(z)F_i(w)=F_i(w)F_i(z)(q^{-1}z-qw)\,, \tag{3.13}$$

$$(q^{-1}z-qw)F_i(z)F_{i+1}(w)=F_{i+1}(w)F_i(z)(z-w)\,, \tag{3.14}$$

$$k^{\pm}_i(z)F_i(w)\left(k^{\pm}_i(z)\right)^{-1}=\frac{q^{-1}z-qw}{z-w}F_i(w)\,, \tag{3.15}$$

$$k^{\pm}_{i+1}(z)F_i(w)\left(k^{\pm}_{i+1}(z)\right)^{-1}=\frac{qz-q^{-1}w}{z-w}F_i(w)\,, \tag{3.16}$$

$$k^{\pm}_i(z)F_j(w)\left(k^{\pm}_i(z)\right)^{-1}=F_j(w),\qquad \text{if}\quad i\ne j,j+1\,,$$

$$[E_i(z),F_j(w)]=\delta_{i,j}\ \delta(z/w)\ (q-q^{-1})\left(k^{+}_i(z)/k^{+}_{i+1}(z)-k^{-}_i(w)/k^{-}_{i+1}(w)\right)\,, \tag{3.17}$$

and the Serre relations for the currents $E_i(z)$ and $F_i(z)$ which are unimportant for this paper.

Commutation relations for the algebra $U_q(\widehat{\mathfrak{gl}}_3)$ given in terms of the currents should be considered as formal series identities describing the infinite set of the relations between modes of these currents. The symbol $\delta(z)$ entering these relations is a formal series $\sum_{n\in\mathbb{Z}} z^n$.

For any series $G(t) = \sum_{m\in\mathbb{Z}} G[m]t^{-m}$ we denote $G(t)^{(+)} = \sum_{m>0} G[m]\, t^{-m}$, and $G(t)^{(-)} = -\sum_{m\le 0} G[m]\, t^{-m}$. Using this notation the Ding–Frenkel formulas (3.8) can be inverted

$$\mathrm{F}^{\pm}_{i+1\,i}(z) = z\left(z^{-1}F_i(z)\right)^{(\pm)}, \qquad \mathrm{E}^{\pm}_{i\,i+1}(z) = E_i(z)^{(\pm)}\,. \tag{3.18}$$

3.2 Different type Borel subalgebras and ordering of current generators

The isomorphism between L-operator [14] and current [13] formulation of the quantum affine algebra proved in [15] allows to express the modes of the L-operators through the modes of the currents and vice versa using the initial relation (3.1) and formulas (3.5)–(3.7). On the other hand, it was proved in [22] that the current generators for the quantum affine algebras form the part of the Cartan–Weyl basis in these algebras.

There exists a natural ordering in the Cartan–Weyl basis. If the generator e_γ corresponds to a positive root $\gamma = \alpha + \beta$, where α and β are roots, then these generators are ordered either in a way $e_\alpha \prec e_\gamma \prec e_\beta$ or in the way $e_\beta \prec e_\gamma \prec e_\alpha$. An important property of the Cartan–Weyl basis of a Borel subalgebra of quantum algebras is that the q-commutator of any two generators from this subalgebra, say e_α and e_β, is a linear combination of monomials containing only the products of generator e_{γ_i} which are 'between' e_α and e_β:

$$e_\alpha \prec e_{\gamma_i} \prec e_\beta \quad \text{or} \quad e_\alpha \succ e_{\gamma_i} \succ e_\beta\,. \tag{3.19}$$

This property of the Cartan–Weyl basis allows one to describe easily the subalgebras in the quantum affine algebras. For instance, in the example above all generators corresponding to the roots α, γ_i, β form a subalgebra by definition. The standard positive Borel subalgebra in $U_q(\widehat{\mathfrak{gl}}_3)$ generated by the modes of L-operators (3.2) is formed by the Cartan–Weyl generators which are 'between' the affine root generator e_{α_0} and non-affine negative simple roots generators $e_{-\alpha_1}$ and $e_{-\alpha_2}$. Respectively, negative Borel subalgebra is formed by the generators which are 'between' $e_{\alpha_1}, e_{\alpha_2}$ and $e_{-\alpha_0}$.

The ordering on the Borel subalgebra can be extended to the ordering of the whole set of Cartan–Weyl generators corresponding to the positive and negative roots such that the same ordering property is valid. This ordering is called 'circular' or 'convex' and it allows one to order arbitrary monomials in the whole algebra [11].

We consider two types of Borel subalgebras of the algebra $U_q(\widehat{\mathfrak{gl}}_3)$. Standard positive and negative Borel subalgebras $U_q(\mathfrak{b}^\pm) \subset U_q(\widehat{\mathfrak{gl}}_3)$ are generated by the modes of the L-operators $L^{(\pm)}(u)$ respectively. For the generators in these subalgebras we can use the modes of the Gauss coordinates (3.5)–(3.7) $\mathrm{E}^{\pm}_{i\,i+1}(u)$, $\mathrm{F}^{\pm}_{i+1\,i}(u)$, $k^{\pm}_j(u)$, $i = 1,2$, $j = 1,2,3$.

Another types of Borel subalgebras are related to the current realizations of $U_q(\widehat{\mathfrak{gl}}_3)$ given in the previous subsection. The Borel subalgebra $U_F \subset U_q(\widehat{\mathfrak{gl}}_3)$ is generated by modes of the currents $F_i[n]$, $k^+_j[m]$, $i = 1,2$, $j = 1,2,3$, $n \in \mathbb{Z}$ and $m \ge 0$. The Borel subalgebra $U_E \subset U_q(\widehat{\mathfrak{gl}}_3)$

is generated by the modes of the currents $E_i[n]$, $k_j^-[-m]$, $i = 1,2$, $j = 1,2,3$, $n \in \mathbb{Z}$ and $m \geq 0$. We will consider also a subalgebras $U'_F = U_F \setminus \{k_j^+[0]\}$ and $U'_E = U_E \setminus \{k_j^-[0]\}$.[3]

Further, we will be interested in the intersections,

$$\begin{aligned} U_F^- &= U'_F \cap U_q(\mathfrak{b}^-)\,, \qquad & U_F^+ &= U_F \cap U_q(\mathfrak{b}^+)\,, \\ U_E^- &= U_E \cap U_q(\mathfrak{b}^-)\,, \qquad & U_E^+ &= U'_E \cap U_q(\mathfrak{b}^+)\,, \end{aligned} \tag{3.20}$$

and will describe properties of projections to these intersections. We call U_F and U_E <u>the current Borel subalgebras</u>. Let $U_f \subset U_F$ and $U_e \subset U_E$ be subalgebras of the current Borel subalgebras generated by the modes of the currents $F_i[n]$ and $E_i[n]$, $i = 1,2$, $n \in \mathbb{Z}$ only. In what follows we will use the subalgebras $U_f^+ \subset U_f$ and $U_e^+ \subset U_e$ defined by the intersections

$$U_f^+ = U_F^+ \cap U_f \qquad U_e^+ = U_E^+ \cap U_e\,. \tag{3.21}$$

Let $U_k^\pm$ be subalgebras in $U_q(\widehat{\mathfrak{gl}}_3)$ generated by the modes of the Cartan currents $k_j^\pm(u)$.

We fix a 'circular' ordering '$\prec$' on the generators of $\overline{U}_q(\mathfrak{gl}_3)$ (see [11]), such that:

$$\cdots \prec U_k^- \prec U_f^- \prec U_f^+ \prec U_k^+ \prec U_e^+ \prec U_e^- \prec U_k^- \prec \cdots\,. \tag{3.22}$$

Ordering of the subalgebras described above can be pictured in the Figure 1 in the anti-clockwise direction.

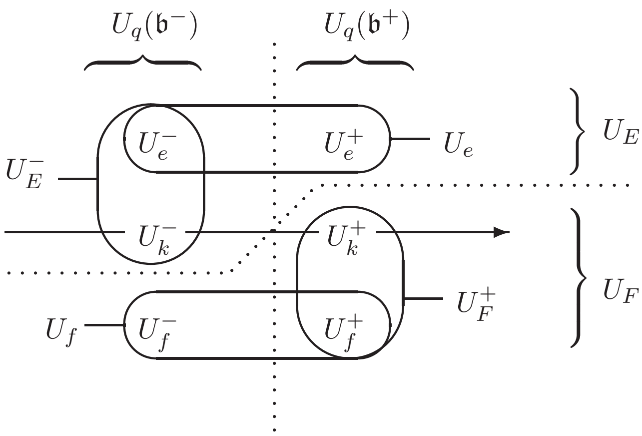

Figure 1: Subalgebras of $\overline{U}_q(\mathfrak{gl}_3)$. Vertical dotted line separates standard Borel subalgebras. Horizontal dotted line separates current Borel subalgebras. Horizontal solid axis shows increasing of the modes of the current generators. Ovals denote different subalgebras in the standard and current Borel subalgebras of $\overline{U}_q(\mathfrak{gl}_3)$.

We will call an element $W \in \overline{U}_q(\mathfrak{gl}_3)$ <u>normal ordered</u> and denote it as $:W:$ if it is presented as linear combinations of products $W_1 \cdot W_2 \cdot W_3 \cdot W_4 \cdot W_5 \cdot W_6$ such that

$$W_1 \in U_f^-\,, \quad W_2 \in U_f^+\,, \quad W_3 \in U_k^+\,, \quad W_4 \in U_e^+\,, \quad W_5 \in U_e^-\,, \quad W_6 \in U_k^-\,.$$

[3] In order to obtain the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_3)$ in the framework of the quantum double construction [13] one has to impose the relation $k_j^+[0]k_j^-[0] = 1$, $j = 1,2,3$.

We may consider standard Borel subalgebras as ordered with respect to the circular ordering (3.22):

$$U_q(\mathfrak{b}^-) = U_e^- \cdot U_k^- \cdot U_f^- \,, \quad U_q(\mathfrak{b}^+) = U_f^+ \cdot U_k^+ \cdot U_e^+ \,.$$

Analogous statement is valid for the current Borel subalgebras:

$$U_F = U_f^- \cdot U_f^+ \cdot U_k^+ \,, \quad U_E = U_e^+ \cdot U_e^- \cdot U_k^- \,.$$

Let us note that the matrix elements in the universal monodromy matrix $L^+(u)$ given by the formulas (3.5)–(3.7) are normal ordered, i.e. $:L^+(u): = L^+(u)$. Now, the problem which we address in this paper, namely the calculation of the action of the monodromy matrix elements onto off-shell Bethe vectors (2.7), can be reformulated as the problem of normal ordering the product of these elements and the element $P_f^+\left(F_2(v_b)\cdots F_2(v_1)\cdot F_1(u_a)\cdots F_1(u_1)\right) \in U_f^+$ modulo terms which annihilate the right vacuum vector $|0\rangle$. Using Gauss decomposition (3.5)–(3.7) this is reduced to the commutation of the Gauss coordinates $E_{ij}^+(u)$ with the element $P_f^+\left(F_2(v_b)\cdots F_2(v_1)\cdot F_1(u_a)\cdots F_1(u_1)\right)$. This way of doing normal ordering is equivalent to the use of the RTT commutation relation and explicit expression for the off-shell Bethe vectors in the form (5.1) and is far too complicated to be useful.

In this paper, we will employe a different and more efficient strategy: we will use the method of projections introduced in [12] and exploited in a series of papers (see [8] and references therein) to relate the off-shell Bethe vectors with the current realization of the quantum affine algebras. We refer the readers to the above mentioned papers to find a complete theory of the projections onto intersections of the different type Borel subalgebras. Here, we will give only some short definition of the projection. In order to do this, we need to equip the algebra $U_q(\widehat{\mathfrak{gl}}_3)$ together with its decomposition into current Borel subalgebras by the current Hopf structure

$$\begin{aligned}
\Delta^{(D)}\left(E_i(z)\right) &= E_i(z)\otimes 1 + k_i^-(z)\left(k_{i+1}^-(z)\right)^{-1}\otimes E_i(z)\,,\\
\Delta^{(D)}\left(F_i(z)\right) &= 1\otimes F_i(z) + F_i(z)\otimes k_i^+(z)\left(k_{i+1}^+(z)\right)^{-1}\,,\\
\Delta^{(D)}\left(k_i^\pm(z)\right) &= k_i^\pm(z)\otimes k_i^\pm(z)\,.
\end{aligned} \tag{3.23}$$

According to the general theory [11] we introduce the projection operators

$$P_f^\pm : U_F \subset U_q(\widehat{\mathfrak{gl}}_3) \to U_F^\pm\,, \qquad P_e^\pm : U_E \subset U_q(\widehat{\mathfrak{gl}}_3) \to U_E^\pm\,.$$

They are respectively defined by the prescriptions

$$P_f^+(f_-\ f_+) = \varepsilon(f_-)\ f_+, \quad P_f^-(f_-\ f_+) = f_-\ \varepsilon(f_+), \quad \forall\, f_- \in U_F^-, \quad \forall\, f_+ \in U_F^+\,, \tag{3.24}$$

$$P_e^+(e_+\ e_-) = e_+\ \varepsilon(e_-), \quad P_e^-(e_-\ e_+) = \varepsilon(e_+)\ e_-, \quad \forall\, e_- \in U_E^-, \quad \forall\, e_+ \in U_E^+\,, \tag{3.25}$$

where the counit map $\varepsilon : U_q(\widehat{\mathfrak{gl}}_3) \to \mathbb{C}$ is defined on current generators as follows

$$\varepsilon(1) = \varepsilon\left(k_j^\pm(u)\right) = 1, \quad \varepsilon\left(E_i(u)\right) = \varepsilon\left(F_i(u)\right) = 0\,. \tag{3.26}$$

Denote by $\overline{U}_F$ and $\overline{U}_E$ the extensions of the algebras U_F and U_E formed by infinite sums of monomials which are ordered products $a_{i_1}[n_1]\cdots a_{i_k}[n_k]$ with $n_1 \le \cdots \le n_k$, where $a_{i_l}[n_l]$ is either $F_{i_l}[n_l]$ or $k_{i_l}^+[n_l]$ and $E_{i_l}[n_l]$ or $k_{i_l}^-[n_l]$, respectively. It can be checked that

(1) the action of the projections (3.24) can be extended to the algebra $\overline{U}_F$;

(2) for any $f \in \overline{U}_F$ with $\Delta^{(D)}(f) = \sum_i f'_i \otimes f''_i$ we have

$$f = \sum_i P_f^-(f''_i) \cdot P_f^+(f'_i); \tag{3.27}$$

(3) the action of the projections (3.25) can be extended to the algebra $\overline{U}_E$;

(4) for any $e \in \overline{U}_E$ with $\Delta^{(D)}(e) = \sum_i e'_i \otimes e''_i$ we have

$$e = \sum_i P_e^+(e'_i) \cdot P_e^-(e''_i). \tag{3.28}$$

The formulas (3.27) and (3.28) are the main technical tools to calculate projections of currents in the formulas (2.7) and (2.8). These formulas allow to present a product of currents in a normal ordered form using projections and the rather simple current Hopf structure (3.23).

Ding–Frenkel isomorphism between L-operator and current realization of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ [15] identifies Gauss coordinates and full currents through formulas (3.8) and (3.18). But there are also higher Gauss coordinates $\mathrm{F}^{\pm}_{ji}(u)$ and $\mathrm{E}^{\pm}_{ij}(u)$ for $j > i+1$ and their relation to the currents was not established in [15]. It is clear that Gauss coordinates $\mathrm{F}^{\pm}_{i+1\,i}(u) = P_f^{\pm}(F_i(u))$ and $\mathrm{E}^{\pm}_{i\,i+1} = P_e^{\pm}(E_i(u))$ are defined by the corresponding projections of the full currents. In [8], special elements from the completed algebras $\overline{U}_F$ and $\overline{U}_E$ were introduced such that their projections yield the corresponding higher Gauss coordinates. These elements were called 'composed' currents. In the case of quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_3)$, there are only two composed currents

$$F_{3,1}(u) \equiv (q-q^{-1})F_1(u)F_2(u)\,, \quad E_{1,3}(u) \equiv (q-q^{-1})E_2(u)E_1(u) \tag{3.29}$$

such that

$$P_f^+(F_{3,1}(u)) = (q-q^{-1})\mathrm{F}^+_{31}(u) \quad P_e^+(E_{1,3}(u)) = (q-q^{-1})\mathrm{E}^+_{13}(u)\,. \tag{3.30}$$

4 Proofs

In what follows we will identify the monodromy matrix $T(u)$ with L-operator $L^+(u) \in U_q(\mathfrak{b}^+)$ from the positive Borel subalgebra of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_3)$.

4.1 The case $\#\bar{w} = 1$

As we have already mentioned our first goal is the proof of the action formulas (2.9)–(2.17) for the single action of monodromy matrix elements onto off-shell Bethe vectors. In this subsection, we perform this calculation using only commutation relations of $U_q(\widehat{\mathfrak{gl}}_3)$ current generators.

4.1.1 Necessary commutation relations

Since the essential part of the off-shell Bethe vectors is concentrated in projection of full current products, we may consider first the action of monodromy elements onto projection of a special product of the full currents.

According to the properties of the projections (3.27) we can present the projection $P_f^+(\mathcal{F}_2(\bar v)\,\mathcal{F}_1(\bar u))$ in the form

$$P_f^+\left(\mathcal{F}_2(\bar v)\mathcal{F}_1(\bar u)\right)=\mathcal{F}_2(\bar v)\mathcal{F}_1(\bar u)-\sum P_f^-\left(\mathcal{F}''\right)\cdot P_f^+\left(\mathcal{F}'\right)\,, \tag{4.1}$$

where the elements $\mathcal{F}'$ and $\mathcal{F}''$ are defined by the coproduct (3.23)

$$\Delta^{(D)}\left(\mathcal{F}_2(\bar v)\mathcal{F}_1(\bar u)\right)=\sum \mathcal{F}'\otimes\mathcal{F}''\,,$$

and in the r.h.s. of (4.1) the number of currents entering the elements $\mathcal{F}'$ is less than the total number of currents in the original product $\mathcal{F}_2(\bar v)\mathcal{F}_1(\bar u)$. Then we may continue replacing $P_f^+(\mathcal{F}')$ by the r.h.s. of (4.1) up to the trivial identity $P_f^+(F_i(w))=F_i(w)-P_f^-(F_i(w))$ to obtain the presentation of $P_f^+(\mathcal{F}_2(\bar v)\mathcal{F}_1(\bar u))$ as a linear combination of terms which are ordered products of negative projections of the currents and the full currents. The idea of calculation of the action of the monodromy elements is to act on this sum first and then apply the projection P_f^+ to the result. It will be shown below that a lot of terms in this sum disappear. Then, it is easy to control the surviving terms.

Let I be the right ideal of $U_q(\widehat{\mathfrak{gl}}_3)$ generated by all elements of the form $F_i[n]\cdot U_q(\widehat{\mathfrak{gl}}_3)$ for $i=1,2$ and $n<0$. We will denote equalities modulo elements in the ideal I by the symbol '$\sim_I$'. Note that this ideal is annihilated by the projection P_f^+.

A useful presentation of the off-shell Bethe vector was proved in the paper [21] using the notion of q-deformed symmetrization (see Corollary 3.6 in this paper). We rewrite this presentation replacing deformed symmetrization by usual symmetrization (with multiplication by a scalar factor). We have[4] [21, 23]

$$\begin{aligned}
P_f^+(\mathcal{F}_2(\bar v)\mathcal{F}_1(\bar u)) &= \mathcal{F}_2(\bar v)\cdot\mathcal{F}_1(\bar u)-\sum_{i=1}^{b}P_f^-\left[F_{3,2}(v_i)\right]\cdot\mathcal{F}_2(\bar v_i)\cdot\mathcal{F}_1(\bar u)\,\frac{\mathsf{f}(v_i,\bar v_{>i})}{\mathsf{f}(\bar v_{>i},v_i)}\\
&-\sum_{i=1}^{a}P_f^-\left[F_{2,1}(u_i)\right]\cdot\mathcal{F}_2(\bar v)\cdot\mathcal{F}_1(\bar u_i)\,\mathsf{f}(\bar v,u_i)\,\frac{\mathsf{f}(u_i,\bar u_{>i})}{\mathsf{f}(\bar u_{>i},u_i)}\\
&-\sum_{\substack{1\le i\le b\\1\le j\le a}}\frac{P_f^-\left[F_{3,1}(u_j)\right]}{q-q^{-1}}\cdot\mathcal{F}_2(\bar v_i)\cdot\mathcal{F}_1(\bar u_j)\,\mathsf{g}(v_i,u_j)v_i\,\mathsf{f}(\bar v_i,u_j)\,\frac{\mathsf{f}(v_i,\bar v_{>i})}{\mathsf{f}(\bar v_{>i},v_i)}\frac{\mathsf{f}(u_j,\bar u_{>j})}{\mathsf{f}(\bar u_{>j},u_j)}+\mathbb{W}
\end{aligned}\tag{4.2}$$

where the elements $\mathbb{W}$ are such that $P_f^+(T_{ij}(w)\cdot\mathbb{W})=0$. Recall that $\bar v_i$ and $\bar u_j$ are the sets $\bar v\setminus\{v_i\}$ and $\bar u\setminus\{u_j\}$. This fact will be checked further using an equivalence

$$T_{ij}(w)\cdot P_f^-\left[F_{k,l}(u)\right]\sim_I\delta_{i,k}\,(q-q^{-1})^{k-l-1}\mathsf{g}(w,u)\,u\;T_{lj}(w)\,, \tag{4.3}$$

[4] The reasons for existence of the presentation (4.2) were explained in the paper [23], where the whole infinite set of the hierarchical relations between $U_q(\widehat{\mathfrak{gl}}_N)$ off-shell Bethe vectors was described in terms of the generating series.

also proved in [21]. Here and in (4.2) the notation $F_{k,l}(u)$, $1 \le l < k \le 3$ is used to denote the simple and 'composed' currents (see (3.29) and discussion on the 'analytical' properties of the composed currents in [8, 21]):

$$F_{2,1}(u) \equiv F_1(u), \quad F_{3,2}(u) \equiv F_2(u), \quad F_{3,1}(u) \equiv (q - q^{-1}) F_1(u) F_2(u) \,.$$

The equivalence (4.3) allows to prove easily that $P_f^+ \left(T_{ij}(w) \cdot \mathbb{W} \right) = 0$ since the elements of $\mathbb{W}$ can be presented in general as $\sum P_f^- \left(F_{c_1,k} \right) \cdot P_f^- \left(F_{c_2,l} \right) \cdot \mathbb{W}'$ with $c_1 > k$ and $c_2 > l$. For example, for $k = l = 1$ and according to (4.3) the action $T_{ij} \cdot P_f^- \left(F_{c_1,1} \right) \cdot P_f^- \left(F_{c_2,1} \right) \cdot \mathbb{W}'$ is proportional to $\delta_{i,c_1} \delta_{1,c_2} = 0$ since $c_2 > 1$. This means that the action of the elements of the monodromy elements onto universal off-shell Bethe vector is defined only by the four terms presented in (4.2). Then, the calculation of this action will be reduced to the commutation of Gauss coordinates entering the monodromy elements (3.4) and the full currents, that is relatively simple.

The calculation of the action of the monodromy elements onto Bethe vector $P_f^+ (\mathcal{F}_2(\bar{v}) \mathcal{F}_1(\bar{u}))$ is decomposed in several steps. First we use formula (4.3) to get rid of the negative projection of the currents and obtain products of the monodromy elements and the full currents. Then we use the explicit expressions of the monodromy matrix elements (3.5)–(3.7) through the Gauss coordinates to calculate the commutation of the Gauss coordinates $\mathrm{E}_{ij}^+(w)$, $k_i^+(w)$ and the full currents, calculating this commutation modulo certain ideals J and K which will be described below. In the next step, we apply the projection P_f^+ to the result of this calculation to restore the structure of the off-shell Bethe vectors, using formula (2.7). Finally, we rewrite the resulting sum of Bethe vectors as a sum over partitions.

To proceed further, we need to know the commutation relations between Gauss coordinates $\mathrm{E}_{ij}^+(w)$ and the full currents $F_i(u)$. To identify $P_f^+ (\mathcal{F}_2(\bar{v}) \mathcal{F}_1(\bar{u}))$ with off-shell Bethe vector we have to act with this element on the right weight singular vector $|0\rangle$. Thus, we can perform the calculations modulo the right ideal J composed from elements $U_q(\widehat{\mathfrak{gl}}_3) \cdot E_i[n]$ for $i = 1, 2$ and $n \ge 0$. Moreover, the commutation relations of $\mathrm{E}_{ij}^+(u)$ with the full currents $F_i(u)$ produce terms containing the negative Cartan currents $k^-(u)$ which can be neglected since they vanish after application of the projection P_f^+. We denote the ideal formed by such elements by K and equalities modulo elements of the ideals J and K will denote by '$\sim_J$' and '$\sim_K$' respectively.

Let us remind that by definition Gauss coordinates $\mathrm{E}_{12}^+(w)$ and $\mathrm{E}_{23}^+(w)$ coincide with the projection of the simple root currents $E_1(w)$ and $E_2(w)$ (see (3.18))

$$\mathrm{E}_{i\,i+1}^+(w) = P_e^+ \left(E_i(w) \right) = \sum_{n>0} E_i[n] w^{-n} = \oint \frac{dt}{w} \, \frac{E_i(t)}{1 - t/w} \,, \quad i = 1, 2 \,, \tag{4.4}$$

where symbol $\oint dt\, g(t)$ means the term g_{-1} of the formal series $g(t) = \sum_{n \in \mathbb{Z}} g_n t^{-n}$ and the rational function $\frac{1}{1-t/w}$ is understood as a series $\sum_{n \ge 0} (t/w)^n$.

The Gauss coordinate $\mathrm{E}_{13}^+(w)$ can expressed through the current generators using the relation between Gauss coordinates

$$(q - q^{-1}) \mathrm{E}_{13}^+(w) = \mathrm{E}_{12}^+[0] \mathrm{E}_{23}^+(w) - q \mathrm{E}_{23}^+(w) \mathrm{E}_{12}^+[0] \,, \tag{4.5}$$

which follows from the RTT relation (3.3) and (4.4)

$$\mathrm{E}^+_{13}(w) = \frac{1}{q-q^{-1}} \oint \frac{dt}{w(1-t/w)}\ \left(E_1[0]E_2(t) - qE_2(t)E_1[0]\right) . \tag{4.6}$$

Then from (3.17) we observe

$$\begin{aligned} &[\mathrm{E}^+_{i\,i+1}(w), F_j(u)] \sim_K \delta_{ij}\mathsf{g}(w,u)u\ \psi^+_i(u), \quad [E_i[0], F_j(u)] \sim_K \delta_{ij}(q-q^{-1})\psi^+_i(u),\\ &\qquad \psi^+_i(u) = k^+_i(u)/k^+_{i+1}(u), \quad i=1,2. \end{aligned} \tag{4.7}$$

Using also one more relation

$$E_1[0]\psi^+_2(w) - q\psi^+_2(w)E_1[0] = (q-q^{-1})\ \psi^+_2(w)\mathrm{E}^+_{12}(w)\,,$$

which follows from (3.11) and (3.12), we may conclude that the action of the Gauss coordinates $\mathrm{E}^+_{ij}(u)$ onto product of the full currents $\mathcal{F}_2(\bar v)\mathcal{F}_1(\bar u)$ is given by the equalities

$$\begin{aligned} \mathrm{E}^+_{13}(w)\cdot\mathcal{F}_2(\bar v)\mathcal{F}_1(\bar u) \sim_{K,J} &\sum_{\substack{1\le i\le b\\ 1\le j\le a}} \mathcal{F}_2(\bar v_i)\mathcal{F}_1(\bar u_j)\psi^+_2(v_i)\psi^+_1(u_j)\\ &\times \mathsf{g}(w,v_i)v_i\ \mathsf{g}(v_i,u_j)u_j\ \mathsf{f}(v_i,\bar u_j)\ \frac{\mathsf{f}(\bar u_{<j},u_j)}{\mathsf{f}(u_j,\bar u_{<j})}\ \frac{\mathsf{f}(\bar v_{<i},v_i)}{\mathsf{f}(v_i,\bar v_{<i})}\,, \end{aligned} \tag{4.8}$$

$$\mathrm{E}^+_{12}(w)\cdot\mathcal{F}_2(\bar v)\mathcal{F}_1(\bar u) \sim_{K,J} \sum_{j=1}^{a} \mathcal{F}_2(\bar v)\mathcal{F}_1(\bar u_j)\psi^+_1(u_j)\mathsf{g}(w,u_j)u_j\ \ \frac{\mathsf{f}(\bar u_{<j},u_j)}{\mathsf{f}(u_j,\bar u_{<j})}\,, \tag{4.9}$$

$$\mathrm{E}^+_{23}(w)\cdot\mathcal{F}_2(\bar v)\mathcal{F}_1(\bar u) \sim_{K,J} \sum_{i=1}^{b} \mathcal{F}_2(\bar v_i)\mathcal{F}_1(\bar u)\psi^+_2(v_i)\mathsf{g}(w,v_i)v_i\ \mathsf{f}(v_i,\bar u)\ \frac{\mathsf{f}(\bar v_{<i},v_i)}{\mathsf{f}(v_i,\bar v_{<i})}\,. \tag{4.10}$$

Now that we have established the action of Gauss coordinates on products of the full current, we can compute the action of the monodromy operators on Bethe vectors.

4.1.2 Calculation of the action

- The action of $T_{13}(w)$

Let us specialize the vector $\mathbb{B}^{a+1,b+1}(w,\bar u;\bar v,w')$ given by the expression (2.7) at the coinciding points $w'=w$. We have

$$\begin{aligned} &\mathbb{B}^{a+1,b+1}(w,\bar u;\bar v,w')|_{w'=w} = \frac{\beta(\bar u|\bar v)}{\mathsf{f}(\bar v,\bar u)}\ \frac{\mathsf{f}(\bar v,w')\mathsf{f}(w,\bar u)}{\mathsf{f}(\bar v,w)\mathsf{f}(w',\bar u)}\ \mathsf{r}_3(\bar v)\mathsf{r}_3(w')\\ &\times\ \frac{w'-w}{qw'-q^{-1}w}P^+_f\left(F_2(v_b)\cdots F_2(v_1)F_2(w')\cdot F_1(w)F_1(u_a)\cdots F_1(u_1)\right)\Big|_{w'=w}|0\rangle\,. \end{aligned} \tag{4.11}$$

Using the commutation relations (3.14)

$$(w'-w)F_2(w')F_1(w) = F_1(w)F_2(w')(qw'-q^{-1}w)\,,$$

the r.h.s. of (4.11) can be written as

$$\begin{gathered}\mathbb{B}^{a+1,b+1}(w,\bar u;\bar v,w)=\frac{\beta(\bar u|\bar v)}{\mathsf{f}(\bar v,\bar u)}\ \mathsf{r}_3(\bar v)\mathsf{r}_3(w)\\ \times P_f^+\left(F_2(v_b)\cdots F_2(v_1)F_1(w)\cdot F_2(w)F_1(u_a)\cdots F_1(u_1)\right)|0\rangle\,.\end{gathered}\tag{4.12}$$

On the other hand, the action of the elements $T_{13}(w)$ according to the property (4.3) is given only by the first term in the r.h.s. of (4.2), namely by the product of the full currents $\mathcal{F}_2(\bar v)\cdot\mathcal{F}_1(\bar u)$, and using explicit form of the matrix element $T_{13}(w)=\mathrm{F}^+_{31}(w)k^+_3(w)$ we can write

$$T_{13}(w)\mathbb{B}^{a,b}(\bar u;\bar v)=\frac{\beta(\bar u|\bar v)}{\mathsf{f}(\bar v,\bar u)}\ \mathsf{r}_3(\bar v)P_f^+\left(\mathrm{F}^+_{31}(w)k^+_3(w)F_2(v_b)\cdots F_2(v_1)\cdot F_1(u_a)\cdots F_1(u_1)\right)|0\rangle\,.\tag{4.13}$$

Taking into account the relation between Gauss coordinate $\mathrm{F}^+_{31}(w)$ and the projection of the composed current $F_{3,1}(w)=(q-q^{-1})F_1(w)F_2(w)$ [8]

$$P_f^+\left(F_{3,1}(w)\right)=(q-q^{-1})\mathrm{F}^+_{31}(w)\quad\text{or}\quad \mathrm{F}^+_{31}(w)=P_f^+\left(F_1(w)F_2(w)\right)\,,$$

property of the projection operator

$$P_f^+\left(P_f^+(A)\cdot B\right)=P_f^+(A\cdot B)\,,\tag{4.14}$$

and the commutation relation

$$F_1(w)F_2(w)k_3(w)\cdot F_2(v)=F_2(v)\cdot F_1(w)F_2(w)k_3(w)\,,$$

we conclude that the r.h.s. of (4.13) is equal to the r.h.s. of (4.12) up to multiplication by $\lambda_2(w)$ and hence the relation (2.9) for $n=1$ is proved. □

- The action of $T_{12}(w)$

Again due to (4.3) the action of the monodromy matrix element $T_{12}(w)$ onto Bethe vector (2.7) is defined by the product of the full currents $\mathcal{F}_2(\bar v)\cdot\mathcal{F}_1(\bar u)$. Taking into account that

$$T_{12}(w)=\mathrm{F}^+_{21}(w)k^+_2(w)+\mathrm{F}^+_{31}(w)k^+_3(w)\mathrm{E}^+_{23}(w)=\mathrm{F}^+_{21}(w)k^+_2(w)+T_{13}(w)E^+_{23}(w)\,,$$

using (4.10) and commutation relations of the Cartan currents $k^+_2(w)$ with the full currents given by (3.15) and (3.16) we obtain

$$\begin{aligned}T_{12}(w)\mathbb{B}^{a,b}(\bar u,\bar v)&=\lambda_2(w)\ \mathsf{f}(\bar v,w)\mathbb{B}^{a+1,b}(w,\{\bar u;\bar v\})\\ &\quad+T_{13}(w)\sum_{i=1}^{b}\mathsf{K}^{(r)}_1(w|v_i)\ \mathsf{f}(\bar v_i,v_i)\ \mathbb{B}^{a,b-1}(\bar u;\bar v_i)\,.\end{aligned}\tag{4.15}$$

In (4.15) we replace the function $\mathsf{g}(w,v_i)v_i$ by the function $\mathsf{K}_1^{(r)}(w|v_i)$ using (2.6), (A.1). In the first term of the r.h.s. of (4.15) we used again property of the projection (4.14) and the commutation relation

$$F_1(w)k_2^+(w)\cdot F_2(v) = F_2(v)\cdot F_1(w)k_2^+(w)\,.$$

Using already calculated action of the matrix element $T_{13}(w)$ onto off-shell Bethe vector we may rewrite (4.15) in the form

$$\begin{aligned} T_{12}(w)\mathbb{B}^{a,b}(\bar u,\bar v) &= \lambda_2(w)\ \mathsf{f}(\bar v,w)\mathbb{B}^{a+1,b}(\{w,\bar u\};\bar v)\\ &+\lambda_2(w)\sum_{i=1}^{b}\mathsf{K}_1^{(r)}(w|v_i)\ \mathsf{f}(\bar v_i,v_i)\ \mathbb{B}^{a+1,b}(\{w,\bar u\};\{w,\bar v_i\})\,, \end{aligned} \tag{4.16}$$

which can be rewritten in the form (2.10) as a sum over partitions of the set $\xi=\{w,\bar v\}\Rightarrow\{\bar\xi_{\rm I},\bar\xi_{\rm II}\}$, for $\#\bar\xi_{\rm I}=1$ since

$$\left.\frac{\mathsf{K}_1^{(l,r)}(w|\bar\xi_{\rm I})}{\mathsf{f}(w,\bar\xi_{\rm I})}\right|_{\bar\xi_{\rm I}=\{w\}}=1\,. \tag{4.17}$$

The action (2.10) for $n=1$ is proved. □

- The action of $T_{23}(w)$

According to (4.3) the action of the monodromy matrix element $T_{23}(w)=\mathrm{F}_{32}^+(w)k_3^+(w)$ will be defined by the first and third terms of the r.h.s. of (4.2) which produces two terms in the action

$$\begin{aligned} T_{23}(w)\mathbb{B}^{a,b}(\bar u,\bar v) &= \lambda_2(w)\ \mathsf{f}(w,\bar u)\mathbb{B}^{a,b+1}(\bar u;\{\bar v,w\})\\ &- T_{13}(w)\sum_{j=1}^{a}\mathsf{g}(w,u_j)u_j\ \mathsf{f}(u_j,\bar u_j)\ \mathbb{B}^{a-1,b}(\bar u_j;\bar v)\,, \end{aligned} \tag{4.18}$$

or

$$\begin{aligned} T_{23}(w)\mathbb{B}^{a,b}(\bar u,\bar v) &= \lambda_2(w)\ \mathsf{f}(w,\bar u)\mathbb{B}^{a,b+1}(\bar u;\{\bar v,w\})\\ &+\lambda_2(w)\sum_{j=1}^{a}\mathsf{K}_1^{(l)}(u_j|w)\ \mathsf{f}(u_j,\bar u_j)\ \mathbb{B}^{a,b+1}(\{\bar u_j,w\};\{\bar v,w\})\,, \end{aligned} \tag{4.19}$$

which due to (4.17) can be rewritten as the sum over partition of the set $\eta=\{w,\bar u\}\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm II}\}$, for $\#\bar\eta_{\rm I}=1$. The action (2.11) for $n=1$ is proved. □

- The action of $T_{22}(w)$

The action of the matrix element

$$T_{22}(w)=k_2^+(w)+\mathrm{F}_{32}^+(w)k_3^+(w)\mathrm{E}_{23}^+(w)$$

onto off-shell Bethe vector (2.7) is defined according to (4.3) by the first and third terms in (4.2) and using (4.10) we obtain

$$\begin{aligned} T_{22}(w)\mathbb{B}^{a,b}(\bar{u},\bar{v}) = \lambda_2(w)\ \mathsf{f}(w,\bar{u})\mathsf{f}(\bar{v},w)\ \mathbb{B}^{a,b}(\bar{u};\bar{v}) \\ + \lambda_2(w)\mathsf{f}(w,\bar{u})\sum_{i=1}^{b}\mathsf{g}(w,v_i)v_i\ \mathsf{f}(\bar{v}_i,v_i)\mathbb{B}^{a,b}(\bar{u};\{\bar{v}_i,w\}) \\ + \sum_{j=1}^{a}\mathsf{g}(u_j,w)u_j\ \mathsf{f}(u_j,\bar{u}_j)\ T_{12}(w)\mathbb{B}^{a-1,b}(\bar{u}_j;\bar{v}) . \end{aligned} \tag{4.20}$$

Using now explicit formula (4.16) for the action of the monodromy matrix element $T_{12}(w)$ onto off-shell Bethe vector we may rewrite (4.20) in the form

$$\begin{aligned} T_{22}(w)\mathbb{B}^{a,b}(\bar{u},\bar{v}) &= \lambda_2(w)\ \mathsf{f}(w,\bar{u})\mathsf{f}(\bar{v},w)\ \mathbb{B}^{a,b}(\bar{u};\bar{v}) \\ &+ \lambda_2(w)\mathsf{f}(w,\bar{u})\sum_{i=1}^{b}\mathsf{K}^{(r)}_1(w|v_i)\ \mathsf{f}(\bar{v}_i,v_i)\mathbb{B}^{a,b}(\bar{u};\{\bar{v}_i,w\}) \\ &+ \lambda_2(w)\mathsf{f}(\bar{v},w)\sum_{j=1}^{a}\mathsf{K}^{(l)}_1(u_j|w)\ \mathsf{f}(u_j,\bar{u}_j)\ \mathbb{B}^{a,b}(\{\bar{u}_j,w\};\bar{v}) \\ &+ \lambda_2(w)\sum_{\substack{1\le i\le b\\ 1\le j\le a}}\mathsf{K}^{(l)}_1(u_j|w)\ \mathsf{K}^{(r)}_1(w|v_i)\ \mathsf{f}(\bar{v}_i,v_i)\ \mathsf{f}(u_j,\bar{u}_j)\ \mathbb{B}^{a,b}(\{\bar{u}_j,w\};\{\bar{v}_i,w\}) , \end{aligned} \tag{4.21}$$

which can be presented as sum over partitions (2.12) of the sets

$$\eta=\{w,\bar{u}\}\Rightarrow\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm II}\}\quad\text{and}\quad \xi=\{w,\bar{v}\}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}\quad\text{for}\quad \#\bar{\eta}_{\rm I}=\#\bar{\xi}_{\rm I}=1. \tag{4.22}$$

The action (2.12) for $n=1$ is proved. □

- The action of $T_{11}(w)$

The action of the matrix element

$$\begin{aligned} T_{11}(w) &= k^+_1(w) + \mathrm{F}^+_{21}(w)k^+_2(w)\mathrm{E}^+_{12}(w) + \mathrm{F}^+_{31}(w)k^+_3(w)\mathrm{E}^+_{13}(w) \\ &= k^+_1(w) + \mathrm{F}^+_{21}(w)k^+_2(w)\mathrm{E}^+_{12}(w) + T_{13}(w)E^+_{13}(w) \end{aligned}$$

as well as the matrix elements $T_{12}(w)$ and $T_{13}(w)$ is defined due to (4.3) by the first term in (4.2). Using formulas (4.8) and (4.9) we obtain

$$\begin{aligned} T_{11}(w)\mathbb{B}^{a,b}(\bar{u},\bar{v}) &= \lambda_2(w)\ \mathsf{r}_1(w)\mathsf{f}(\bar{u},w)\ \mathbb{B}^{a,b}(\bar{u};\bar{v}) \\ &+ \lambda_2(w)\sum_{j=1}^{a}\mathsf{r}_1(u_j)\mathsf{K}^{(r)}_1(w|u_j)\ \frac{\mathsf{f}(\bar{u}_j,u_j)\mathsf{f}(\bar{v},w)}{\mathsf{f}(\bar{v},u_j)}\ \mathbb{B}^{a,b}(\{\bar{u}_j,w\};\bar{v}) \\ &+ \lambda_2(w)\sum_{\substack{1\le i\le b\\ 1\le j\le a}}\mathsf{r}_1(u_j)\mathsf{K}^{(r)}_1(w|v_i)\ \mathsf{K}^{(r)}_1(v_i|u_j)\ \frac{\mathsf{f}(\bar{u}_j,u_j)\mathsf{f}(\bar{v}_i,v_i)}{\mathsf{f}(\bar{v},u_j)}\ \mathbb{B}^{a,b}(\{\bar{u}_j,w\};\{\bar{v}_i,w\}) . \end{aligned} \tag{4.23}$$

The expression (4.23) can be written as a sum (2.13) over partitions (4.22) because the term corresponding to the partition $\xi_{\rm II}=\{\bar v_i,w\}$ and $\eta_{\rm I}=\{w\}$ vanishes due to presence in the denominator of (2.13) the factor $\mathsf{f}(\xi_{\rm II},\eta_{\rm I})$ and the other three type of partitions $\xi_{\rm I}=\eta_{\rm I}=\{w\}$; $\xi_{\rm I}=\{w\}$, $\eta_{\rm I}=\{u_j\}$; $\xi_{\rm I}=\{v_i\}$, $\eta_{\rm I}=\{u_j\}$ yield exactly the three terms in (4.23) due to (4.17). The action (2.13) for $n=1$ is proved. □

- The action of $T_{33}(w)$

According to (4.3) this action will be defined by the first, second and forth terms in (4.2). Using these relations, definition of the universal off-shell Bethe vector (2.7) and the fact that $T_{33}(w)=k_3^+(w)$ we obtain

$$\begin{aligned}
&T_{33}(w)\mathbb{B}^{a,b}(\bar u,\bar v)=\lambda_2(w)\ \mathsf{r}_3(w)\mathsf{f}(w,\bar v)\ \mathbb{B}^{a,b}(\bar u;\bar v)\\
&+\lambda_2(w)\sum_{i=1}^{b}\mathsf{r}_3(v_i)\mathsf{K}_1^{(l)}(v_i|w)\ \frac{\mathsf{f}(v_i,\bar v_i)\mathsf{f}(w,\bar u)}{\mathsf{f}(v_i,\bar u)}\ \mathbb{B}^{a,b}(\bar u;\{\bar v_i,w\})\\
&+\lambda_2(w)\sum_{\substack{1\le i\le b\\ 1\le j\le a}}\mathsf{r}_3(v_i)\mathsf{K}_1^{(l)}(u_j|w)\ \mathsf{K}_1^{(l)}(v_i|u_j)\ \frac{\mathsf{f}(v_i,\bar v_i)\mathsf{f}(u_j,\bar u_j)}{\mathsf{f}(v_i,\bar u)}\ \mathbb{B}^{a,b}(\{\bar u_j,w\};\{\bar v_i,w\})\,.
\end{aligned}\tag{4.24}$$

The expression (4.24) can be written as sum (2.14) over partitions (4.22) because the term corresponding to the partition $\eta_{\rm II}=\{\bar u_j,w\}$ and $\xi_{\rm I}=\{w\}$ vanishes due to presence in the denominator of (2.14) the factor $\mathsf{f}(\xi_{\rm I},\eta_{\rm II})$ and the other three type of partitions $\xi_{\rm I}=\eta_{\rm I}=\{w\}$; $\xi_{\rm I}=\{v_i\}$, $\eta_{\rm I}=\{w\}$; $\xi_{\rm I}=\{v_i\}$, $\eta_{\rm I}=\{u_j\}$ yield exactly the three terms in (4.24) due to (4.17). The action (2.14) for $n=1$ is proved. □

Before continuing with the action of the lower-triangular monodromy matrix entries $T_{21}(w)$, $T_{32}(w)$ and $T_{31}(w)$ onto off shell Bethe vectors, let us check the formulas (4.21), (4.23) and (4.24). It is easy to see that these formula lead to the Bethe equations when one requires that the vector $\mathbb{B}^{a,b}(\bar u;\bar v)$ is an eigenvector of the transfer matrix. Indeed

$$(T_{11}(w)+T_{22}(w)+T_{33}(w))\mathbb{B}^{a,b}(\bar u;\bar v)=\tau(w;\bar u,\bar v)\mathbb{B}^{a,b}(\bar u;\bar v)\,,$$

where

$$\tau(w;\bar u,\bar v)=\lambda_1(w)\mathsf{f}(\bar u,w)+\lambda_2(w)\mathsf{f}(w,\bar u)\mathsf{f}(\bar v,w)+\lambda_3(w)\mathsf{f}(w,\bar v)\,,$$

provided the Bethe equations

$$\mathsf{r}_1(u_j)=\frac{\mathsf{f}(u_j,\bar u_j)}{\mathsf{f}(\bar u_j,u_j)}\ \mathsf{f}(\bar v,u_j),\qquad \mathsf{r}_3(v_i)=\frac{\mathsf{f}(\bar v_i,v_i)}{\mathsf{f}(v_i,\bar v_i)}\ \mathsf{f}(v_i,\bar u)$$

are satisfied. The coefficient of $\mathbb{B}^{a,b}(\{\bar u_j,w\};\{\bar v_i,w\})$ vanishes due to the trivial identity

$$\mathsf{K}_1^{(r)}(w|v_i)\mathsf{K}_1^{(r)}(v_i|u_j)+\mathsf{K}_1^{(l)}(u_j|w)\mathsf{K}_1^{(r)}(w|v_i)+\mathsf{K}_1^{(l)}(u_j|w)\mathsf{K}_1^{(l)}(v_i|u_j)=0.\tag{4.25}$$

We now compute the action of the lower-diagonal monodromy matrix elements onto off-shell Bethe vectors. Let us repeat once again the strategy of our calculation, for example, in the case of the action of the element

$$T_{21}(w) = k_2^+(w)\mathrm{E}_{12}^+(w) + \mathrm{F}_{32}^+(w)k_3^+(w)\mathrm{E}_{13}^+(w)\,.$$

Calculation of the action in our approach means the normal ordering of the product

$$T_{21}(w) \cdot P_f^+\left(F_2(v_b)\cdots F_2(v_1)\cdot F_1(u_a)\cdots F_1(u_1)\right) \tag{4.26}$$

in the context of circular ordering of the Cartan–Weyl or current generators of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_3)$ described in subsection 3.2 and keeping after this ordering only those terms that belong to the subalgebra U_F^+. According to the presentation (4.2) and the equivalence (4.3) the r.h.s. of (4.26) can be written as follows

$$P_f^+\left(T_{21}(w)\cdot\mathcal{F}_2(\bar v)\mathcal{F}_1(\bar u) - \sum_{j=1}^{a}\mathsf{g}(w,u_j)u_j\ \mathsf{f}(\bar v,u_j)T_{11}(w)\cdot\mathcal{F}_2(\bar v)\mathcal{F}_1(\bar u_j)\frac{\mathsf{f}(u_j,\bar u_{>j})}{\mathsf{f}(\bar u_{>j},u_j)}\right)\,, \tag{4.27}$$

where first we calculate the ordering under projection in (4.27) modulo elements from the ideal J and then apply projection only to those terms which do not belong to this ideal. We can simply remove all elements from the ideal J in (4.27) before taking the projection since by definition $J|0\rangle = 0$. Once it is done, we multiply (4.26) and (4.27) by the product $\beta(\bar u|\bar v)\mathsf{r}_3(\bar v)\mathsf{f}^{-1}(\bar v,\bar u)$ and act by both of these elements onto right vacuum vector $|0\rangle$ according to definition (2.7) to recover the action $T_{21}(w)$ onto $\mathbb{B}^{a,b}(\bar u;\bar v)$.

Due to the fact that the matrix elements $T_{1\ell}(w)$, $\ell = 1,2,3$ acts effectively only on the first term in (4.2) we may formally write

$$T_{1\ell}(w)\cdot P_f^+\left(\mathcal{F}_2(\bar v)\cdot\mathcal{F}_1(\bar u)\right) = P_f^+\left(T_{1\ell}(w)\cdot\mathcal{F}_2(\bar v)\cdot\mathcal{F}_1(\bar u)\right) \tag{4.28}$$

understanding this equality in the described above sense. It means that recovering the Bethe vectors in (4.27) we may first interchange the projection P_f^+ and the action of $T_{11}(w)$, then restore the Bethe vector from the projection and then use already calculated action of the monodromy matrix element $T_{11}(w)$ onto $\mathbb{B}^{a,b}(\bar u;\bar v)$ given by (4.23). This will slightly simplify the whole calculation, although we cannot do the same trick for the calculation of the rest matrix elements $T_{ij}(w)$, $i \neq 1$. To calculate the action of these matrix elements onto off-shell Bethe vectors we have to use an explicit expression in terms of the Gauss coordinates and the commutation relations of the Gauss coordinates with the full currents.

- The action of $T_{21}(w)$

Taking these rules into account and using (4.8) and (4.9) we may calculate

$$
\begin{aligned}
T_{21}(w)\mathbb{B}^{a,b}(\bar u;\bar v) = \lambda_2(w)\Bigg(&\sum_{j=1}^{a} \mathsf{K}^{(r)}_1(w|u_j)\ \mathsf{r}_1(u_j)\ \frac{\mathsf{f}(w,\bar u_j)\mathsf{f}(\bar u_j,u_j)\mathsf{f}(\bar v,w)}{\mathsf{f}(\bar v,u_j)}\mathbb{B}^{a-1,b}(\bar u_j;\bar v)\\
&+\sum_{\substack{1\le i\le b\\ 1\le j\le a}} \mathsf{K}^{(r)}_1(w|v_i)\ \mathsf{K}^{(r)}_1(v_i|u_j)\ \mathsf{r}_1(u_j)\ \frac{\mathsf{f}(w,\bar u_j)\mathsf{f}(\bar u_j,u_j)\mathsf{f}(\bar v_i,v_i)}{\mathsf{f}(\bar v,u_j)}\mathbb{B}^{a-1,b}(\bar u_j;\{\bar v_i,w\})\Bigg) \qquad (4.29)\\
&+T_{11}(w)\ \sum_{j=1}^{a}\mathsf{K}^{(l)}_1(u_j|w)\ \mathsf{f}(u_j,\bar u_j)\mathbb{B}^{a-1,b}(\bar u_j;\bar v)\,.
\end{aligned}
$$

Then, using (4.23) the expression (4.29) can be written in the form (2.15) with a sum over partitions of the sets $\bar\eta = \{\bar u, w\} \Rightarrow \{\bar\eta_{\rm I},\bar\eta_{\rm II},\bar\eta_{\rm III}\}$ and $\bar\xi = \{\bar v, w\} \Rightarrow \{\bar\xi_{\rm I},\bar\xi_{\rm II}\}$ such that $\#\bar\eta_{\rm I} = \#\bar\eta_{\rm II} = \#\bar\xi_{\rm I} = 1$. Note that in doing so, one possible partition $\bar\xi_{\rm I} = \{v_i\}$, $\bar\xi_{\rm II} = \{\bar v_i, w\}$, $\bar\eta_{\rm I} = \{w\}$, $\bar\eta_{\rm II} = \{u_j\}$, $\bar\eta_{\rm III} = \{\bar u_j\}$ yields zero contribution due to the factor $\mathsf{f}^{-1}(\bar\xi_{\rm II},\bar\eta_{\rm I})$. The action (2.15) for $n = 1$ is proved. □

- The action of $T_{32}(w)$

Repeating the same arguments we may present the intermediate result for the action of this matrix element

$$
\begin{aligned}
T_{32}(w)\mathbb{B}^{a,b}(\bar u;\bar v) = \lambda_2(w)\Bigg(&\sum_{i=1}^{b} \mathsf{K}^{(r)}_1(w|v_i)\ \mathsf{r}_3(w)\ \frac{\mathsf{f}(w,\bar v_i)\mathsf{f}(\bar v_i,v_i)\mathsf{f}(w,\bar u)}{\mathsf{f}(w,\bar u)}\mathbb{B}^{a,b-1}(\bar u;\bar v_i)\\
&+\sum_{i=1}^{b}\mathsf{K}^{(l)}_1(v_i|w)\ \mathsf{r}_3(v_i)\ \frac{\mathsf{f}(v_i,\bar v_i)\mathsf{f}(\bar v_i,w)\mathsf{f}(w,\bar u)}{\mathsf{f}(v_i,\bar u)}\mathbb{B}^{a,b-1}(\bar u;\bar v_i)\\
&+\sum_{1\le i\ne i'\le b}\mathsf{K}^{(l)}_1(v_i|w)\ \mathsf{K}^{(r)}_1(w|v_{i'})\ \mathsf{r}_3(v_i)\ \frac{\mathsf{f}(v_i,\bar v_i)\mathsf{f}(\bar v_{i,i'},v_{i'})\mathsf{f}(w,\bar u)}{\mathsf{f}(v_i,\bar u)}\mathbb{B}^{a,b-1}(\bar u;\{\bar v_{i,i'},w\})\Bigg) \qquad (4.30)\\
&+T_{12}(w)\sum_{\substack{1\le j\le a\\ 1\le i\le b}}\mathsf{K}^{(l)}_1(u_j|w)\ \mathsf{K}^{(l)}_1(v_i|u_j)\ \mathsf{r}_3(v_i)\ \frac{\mathsf{f}(v_i,\bar v_i)\mathsf{f}(u_j,\bar u_j)}{\mathsf{f}(v_i,\bar u)}\mathbb{B}^{a-1,b-1}(\bar u_j;\bar v_i)
\end{aligned}
$$

Using (4.16) we may present (4.30) in the form (2.16) as sum over partitions of the sets $\bar\eta = \{\bar u, w\} \Rightarrow \{\bar\eta_{\rm I},\bar\eta_{\rm II}\}$ and $\bar\xi = \{\bar v, w\} \Rightarrow \{\bar\xi_{\rm I},\bar\xi_{\rm II},\bar\xi_{\rm III}\}$ such that $\#\bar\xi_{\rm I} = \#\bar\xi_{\rm II} = \#\bar\eta_{\rm I} = 1$. The action (2.16) for $n = 1$ is proved. □

- The action of $T_{31}(w)$

The action of the matrix element $T_{31}(w)$ can be calculated analogously. The intermediate result of this action is

$$
\begin{aligned}
&T_{31}(w)\mathbb{B}^{a,b}(\bar u;\bar v)\\
&=\lambda_2(w)\Biggl(\sum_{\substack{1\le j\le a\\1\le i\le b}}\mathsf{K}_1^{(r)}(v_i|u_j)\ \mathsf{K}_1^{(r)}(w|v_i)\ \mathsf{r}_1(u_j)\mathsf{r}_3(w)\frac{\mathsf{f}(\bar u_j,u_j)\mathsf{f}(w,\bar v_i)\mathsf{f}(\bar v_i,v_i)}{\mathsf{f}(\bar v,u_j)}\ \mathbb{B}^{a-1,b-1}(\bar u_j;\bar v_i)\\
&+\sum_{\substack{1\le j\le a\\1\le i\le b}}\mathsf{K}_1^{(l)}(v_i|w)\ \mathsf{K}_1^{(r)}(w|u_j)\ \mathsf{r}_1(u_j)\mathsf{r}_3(v_i)\ \frac{\mathsf{f}(\bar u_j,u_j)\mathsf{f}(w,\bar u_j)\mathsf{f}(v_i,\bar v_i)\mathsf{f}(\bar v_i,w)}{\mathsf{f}(v_i,u_j)\mathsf{f}(v_i,\bar u_j)\mathsf{f}(\bar v_i,u_j)}\ \mathbb{B}^{a-1,b-1}(\bar u_j;\bar v_i)\\
&+\sum_{\substack{1\le j\le a\\1\le i\ne i'\le b}}\mathsf{K}_1^{(l)}(v_i|w)\mathsf{K}_1^{(r)}(v_{i'}|u_j)\ \mathsf{K}_1^{(r)}(w|v_{i'})\ \mathsf{r}_1(u_j)\mathsf{r}_3(v_i)\\
&\qquad\qquad\times\frac{\mathsf{f}(\bar u_j,u_j)\mathsf{f}(w,\bar u_j)\mathsf{f}(v_i,\bar v_i)\mathsf{f}(\bar v_{i,i'},v_{i'})}{\mathsf{f}(v_i,u_j)\mathsf{f}(v_i,\bar u_j)\mathsf{f}(\bar v_i,u_j)}\ \mathbb{B}^{a-1,b-1}(\bar u_j;\{\bar v_{i,i'},w\})\Biggr)\\
&+T_{11}(w)\sum_{\substack{1\le j\le a\\1\le i\le b}}\mathsf{K}_1^{(l)}(v_i|u_j)\ \mathsf{K}_1^{(l)}(u_j|w)\ \mathsf{r}_3(v_i)\frac{\mathsf{f}(u_j,\bar u_j)\mathsf{f}(v_i,\bar v_i)}{\mathsf{f}(v_i,\bar u)}\ \mathbb{B}^{a-1,b-1}(\bar u_j;\bar v_i)\,.
\end{aligned}
\tag{4.31}
$$

Using (4.23) we conclude that the final result of the action of the monodromy matrix elements $T_{31}(w)$ can be written in the form (2.17) as sum over partitions of the sets $\bar\eta=\{\bar u,w\}\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm II},\bar\eta_{\rm III}\}$ and $\bar\xi=\{\bar v,w\}\Rightarrow\{\bar\xi_{\rm I},\bar\xi_{\rm II},\bar\xi_{\rm III}\}$ such that $\#\bar\xi_{\rm I}=\#\bar\xi_{\rm II}=\#\bar\eta_{\rm I}=\#\bar\eta_{\rm II}=1$. The action (2.17) for $n=1$ is proved. □

4.2 The general case $\#\bar w=n$

We have proved the formulas of multiple actions (2.9)–(2.17) for $\#\bar w=1$. Then the general case $\#\bar w=n$ can be considered via induction over n. We assume that the equations (2.9)–(2.17) are valid for $\#\bar w=n-1$ and act successively: first by $T_{ij}(\bar w_n)$ and then by $T_{ij}(w_n)$. The induction for (2.9) is trivial. The proofs of the other formulas require the use of lemma A.1.

Consider, for instance, the multiple action of $T_{23}(\bar w)$. It is convenient to write (2.11) in the following form:

$$
T_{23}(\bar w_n)\mathbb{B}^{a,b}(\bar u;\bar v)=(-q)^{1-n}\lambda_2(\bar w_n)\sum_{\{\bar w_n,\bar u\}\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm II}\}}\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm II})\mathsf{K}_{n-1}^{(r)}(\bar w_nq^{-2}|\bar\eta_{\rm I})\ \mathbb{B}^{a,b+n-1}(\bar\eta_{\rm II};\bar\xi)\,.
\tag{4.32}
$$

Here we have got rid of the poles of $\mathsf{K}_{n-1}^{(l)}(\bar\eta_{\rm I}|\bar w_n)$ at $\eta_i=w_j$ transforming it into $\mathsf{K}_{n-1}^{(r)}(\bar w_nq^{-2}|\bar\eta_{\rm I})$ via (A.6). Thus, the action of $T_{23}(\bar w_n)$ produces the sum over partitions of the set $\{\bar w_n,\bar u\}$ into

subsets $\bar\eta_{\rm I}$ and $\bar\eta_{\rm II}$. Applying the operator $T_{23}(w_n)$ to (4.32) we obtain

$$T_{23}(\bar w)\mathbb{B}^{a,b}(\bar u;\bar v)=(-q)^{-n}\lambda_2(\bar w)\sum_{\{\bar w_n,\bar u\}\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm II}\}}\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm II})\mathsf{K}^{(r)}_{n-1}(\bar w_n q^{-2}|\bar\eta_{\rm I})$$
$$\times\sum_{\{w_n,\bar\eta_{\rm II}\}\Rightarrow\{\bar\eta_{\rm i},\bar\eta_{\rm ii}\}}\mathsf{f}(\bar\eta_{\rm i},\bar\eta_{\rm ii})\mathsf{K}^{(r)}_{1}(w_n q^{-2}|\bar\eta_{\rm i})\;\mathbb{B}^{a,b+n}(\bar\eta_{\rm ii};\bar\xi)\,. \tag{4.33}$$

Here we have an additional sum over partitions of the set $\{w_n,\bar\eta_{\rm II}\}$ into subsets $\bar\eta_{\rm i}$ and $\bar\eta_{\rm ii}$. In fact, one can say that we have the sum over partitions of the set $\{\bar w,\bar u\}$ into three subsets $\bar\eta_{\rm I}$, $\bar\eta_{\rm i}$, and $\bar\eta_{\rm ii}$ with one additional constraint $w_n\notin\bar\eta_{\rm I}$.

Obviously

$$\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm II})=\frac{\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm II})\mathsf{f}(\bar\eta_{\rm I},w_n)}{\mathsf{f}(\bar\eta_{\rm I},w_n)}=\frac{\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm i})\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm ii})}{\mathsf{f}(\bar\eta_{\rm I},w_n)}\,. \tag{4.34}$$

It is easy to see that the function in the r.h.s. of (4.34) is a projector of the product $\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm II})$ onto partitions $\bar\eta_{\rm I}$, $\bar\eta_{\rm i}$, and $\bar\eta_{\rm ii}$, such that $w_n\notin\bar\eta_{\rm I}$:

$$\frac{\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm i})\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm ii})}{\mathsf{f}(\bar\eta_{\rm I},w_n)}=\begin{cases}\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm II})\,, & \text{if}\quad w_n\notin\bar\eta_{\rm I}\,,\\ 0\,, & \text{if}\quad w_n\in\bar\eta_{\rm I}\,.\end{cases} \tag{4.35}$$

Then the sum (4.33) takes the form

$$T_{23}(\bar w)\mathbb{B}^{a,b}(\bar u;\bar v)=(-q)^{-n}\lambda_2(\bar w)\sum_{\{\bar w,\bar u\}\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm i},\bar\eta_{\rm ii}\}}\mathsf{K}^{(r)}_{n-1}(\bar w_n q^{-2}|\bar\eta_{\rm I})\mathsf{K}^{(r)}_{1}(w_n q^{-2}|\bar\eta_{\rm i})$$
$$\times\frac{\mathsf{f}(\bar\eta_{\rm i},\bar\eta_{\rm ii})\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm i})\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm ii})}{\mathsf{f}(\bar\eta_{\rm I},w_n)}\;\mathbb{B}^{a,b+n}(\bar\eta_{\rm ii};\bar\xi)\,. \tag{4.36}$$

Setting $\{\bar\eta_{\rm I},\bar\eta_{\rm i}\}=\bar\eta_0$ and transforming $\mathsf{K}^{(r)}_{1}(w_n q^{-2}|\bar\eta_{\rm i})$ via (A.6) we obtain

$$T_{23}(\bar w)\mathbb{B}^{a,b}(\bar u;\bar v)=(-q)^{1-n}\lambda_2(\bar w)\sum_{\{\bar w,\bar u\}\Rightarrow\{\bar\eta_0,\bar\eta_{\rm ii}\}}\frac{\mathsf{f}(\bar\eta_0,\bar\eta_{\rm ii})}{\mathsf{f}(\bar\eta_0,w_n)}\;\mathbb{B}^{a,b+n}(\bar\eta_{\rm ii};\bar\xi)$$
$$\times\sum_{\bar\eta_0\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm i}\}}\mathsf{K}^{(l)}_{1}(\bar\eta_{\rm i}|w_n)\mathsf{K}^{(r)}_{n-1}(\bar w_n q^{-2}|\bar\eta_{\rm I})\mathsf{f}(\bar\eta_{\rm I},\bar\eta_{\rm i}). \tag{4.37}$$

The sum over partitions $\bar\eta_0\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm i}\}$ in the last line of (4.37) can be computed via (A.9), what gives us

$$T_{23}(\bar w)\mathbb{B}^{a,b}(\bar u;\bar v)=\lambda_2(\bar w)\sum_{\{\bar w,\bar u\}\Rightarrow\{\bar\eta_0,\bar\eta_{\rm ii}\}}\frac{\mathsf{f}(\bar\eta_0,\bar\eta_{\rm ii})\mathsf{f}(\bar w_n q^{-2},\bar\eta_0)}{\mathsf{f}(\bar\eta_0,w_n)}\mathsf{K}^{(l)}_{n}(\bar\eta_0|\bar w)\;\mathbb{B}^{a,b+n}(\bar\eta_{\rm ii};\bar\xi)\,. \tag{4.38}$$

It remains to use $\mathsf{f}(\bar w_n q^{-2},\bar\eta_0)=\mathsf{f}^{-1}(\bar\eta_0,\bar w_n)$, and we arrive at (2.11) with $\#\bar w=n$. □

All other formulas of multiple actions are proved in exactly the same manner. Successive action of $T_{ij}(\bar w_n)$ and $T_{ij}(w_n)$ gives a sum over partitions with constraints. Introducing appropriate projectors as in (4.35) we get rid of these constraints. Then certain sums over partitions can be computed via lemma A.1. The details of these calculations, however, are rather cumbersome, therefore we do not give them here.

5 Conclusion

The explicit formulas for the monodromy matrix elements acting onto off-shell nested Bethe vectors hopefully will help to calculate form factors of local operators, in the framework of the approach developed in [24]. As in the case of rational $SU(3)$-symmetric quantum integrable models [25], it will also lead to formula for the scalar products of the off-shell nested Bethe vectors in quantum integrable models with $GL(3)$ trigonometric R-matrix. Indeed, the off-shell Bethe vectors given by formulas (2.7) and (2.8) can be rewritten through the matrix elements of the monodromy[5] (see also [8, 9]):

$$\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \sum \frac{\mathsf{K}_k^{(r)}(\bar{v}_{\rm I}|\bar{u}_{\rm I})}{\lambda_2(\bar{u}_{\rm II})\lambda_2(\bar{v})} \frac{\mathsf{f}(\bar{v}_{\rm II},\bar{v}_{\rm I})\mathsf{f}(\bar{u}_{\rm I},\bar{u}_{\rm II})}{\mathsf{f}(\bar{v},\bar{u})} T_{13}(\bar{v}_{\rm I})T_{23}(\bar{v}_{\rm II})T_{12}(\bar{u}_{\rm II})|0\rangle\,, \tag{5.1}$$

$$\mathbb{C}^{a,b}(\bar{u};\bar{v}) = \sum \frac{\mathsf{K}_k^{(l)}(\bar{v}_{\rm I}|\bar{u}_{\rm I})}{\lambda_2(\bar{u}_{\rm II})\lambda_2(\bar{v})} \frac{\mathsf{f}(\bar{v}_{\rm II},\bar{v}_{\rm I})\mathsf{f}(\bar{u}_{\rm I},\bar{u}_{\rm II})}{\mathsf{f}(\bar{v},\bar{u})} \langle 0|\, T_{21}(\bar{u}_{\rm II})T_{32}(\bar{v}_{\rm II})T_{31}(\bar{v}_{\rm I})\,, \tag{5.2}$$

where the sum goes over all partitions of the sets $\bar{u} \Rightarrow \{\bar{u}_{\rm I},\bar{u}_{\rm II}\}$ and $\bar{v} \Rightarrow \{\bar{v}_{\rm I},\bar{v}_{\rm II}\}$ such that $\#\bar{u}_{\rm I} = \#\bar{v}_{\rm I} = k$, $k = 0,\dots,\min(a,b)$. The proof of formulas (5.1) and (5.2) will be given elsewhere. In principle, one can use these formulas to prove the relations (2.9)–(2.17) using multiple exchange relations and the properties of the Izergin determinant as it was done in [26] for the $GL(3)$-invariant integrable models associated with rational R-matrix. However, we showed in this paper that the use of current presentation provides a simpler way to perform the calculation.

Combining the explicit presentations (5.1) and (5.2) with the multiple actions calculated in the present paper, we can hope to tackle the problem of computing form factors and scalar products. This strategy was applied successfully to the case of $GL(3)$-invariant integrable models associated with rational R-matrix, giving some hope for the trigonometric case.

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A Properties of Izergin determinant

The following properties of Izergin determinant easily follows from the definition (2.5).

Initial condition:

$$\mathsf{K}_1(\bar{x}|\bar{y}) = \mathsf{g}(x,y)\,. \tag{A.1}$$

[5] Observe that up to replacement $\mathsf{K}_k^{(l,r)} \to \mathsf{K}_k$ these formulas have the same structure as the formulas for Bethe vectors in rational $GL(3)$-invariant models.

Rescaling of the arguments:

$$\mathsf{K}_n(\alpha\bar{x}|\alpha\bar{y}) = \alpha^{-n}\mathsf{K}_n(\bar{x}|\bar{y}q^2)\,. \tag{A.2}$$

Reduction:

$$\begin{aligned} \mathsf{K}_n(\bar{x}, zq^{-2}|\bar{y}, z) &= -\frac{q}{z}\mathsf{K}_n(\bar{x}|\bar{y})\,, \\ \mathsf{K}_n(\bar{x}, z|\bar{y}, zq^2) &= -\frac{1}{qz}\mathsf{K}_n(\bar{x}|\bar{y})\,. \end{aligned} \tag{A.3}$$

Inverse order of arguments:

$$\begin{aligned} \mathsf{K}_n(\bar{x}q^{-2}|\bar{y}) &= (-q)^n\mathsf{f}^{-1}(\bar{y}, \bar{x})\mathsf{K}_n(\bar{y}|\bar{x})\,, \\ \mathsf{K}_n(\bar{x}|\bar{y}q^2) &= (-q)^{-n}\mathsf{f}^{-1}(\bar{y}, \bar{x})\mathsf{K}_n(\bar{y}|\bar{x})\,. \end{aligned} \tag{A.4}$$

Residues in the poles at $x_j = y_k$:

$$\mathsf{K}_n(\bar{x}|\bar{y})|_{x_n\to y_n} = \mathsf{g}(x_n, y_n)\mathsf{f}(y_n, \bar{y}_n)\mathsf{f}(\bar{x}_n, x_n)\mathsf{K}_{n-1}(\bar{x}_n|\bar{y}_n) + \mathrm{reg}\,, \tag{A.5}$$

where reg means regular part.

Using these properties of K_n one can easily derive similar properties for its modifications $\mathsf{K}^{(l,r)}$, in particular,

$$\begin{aligned} \mathsf{K}^{(r)}_n(\bar{x}q^{-2}|\bar{y}) &= (-q)^n\mathsf{f}^{-1}(\bar{y}, \bar{x})\mathsf{K}^{(l)}_n(\bar{y}|\bar{x})\,, \\ \mathsf{K}^{(l)}_n(\bar{x}|\bar{y}q^2) &= (-q)^{-n}\mathsf{f}^{-1}(\bar{y}, \bar{x})\mathsf{K}^{(r)}_n(\bar{y}|\bar{x})\,. \end{aligned} \tag{A.6}$$

One more important property of $\mathsf{K}_n(\bar{x}|\bar{y})$ is a summation formula.

Lemma A.1. (Main Lemma) *Let $\bar{\gamma}$, $\bar{\alpha}$ and $\bar{\beta}$ be sets of complex variables with $\#\alpha = m_1$, $\#\beta = m_2$, and $\#\gamma = m_1 + m_2$. Then*

$$\sum \mathsf{K}_{m_1}(\bar{\gamma}_{\rm I}|\bar{\alpha})\mathsf{K}_{m_2}(\bar{\beta}|\bar{\gamma}_{\rm II})\mathsf{f}(\bar{\gamma}_{\rm II}, \bar{\gamma}_{\rm I}) = (-q)^{-m_1}\mathsf{f}(\bar{\gamma}, \bar{\alpha})\mathsf{K}_{m_1+m_2}(\{\bar{\alpha}q^{-2}, \bar{\beta}\}|\bar{\gamma}). \tag{A.7}$$

The sum is taken with respect to all partitions of the set $\bar{\gamma} \Rightarrow \{\bar{\gamma}_{\rm I}, \bar{\gamma}_{\rm II}\}$ with $\#\bar{\gamma}_{\rm I} = m_1$ and $\#\bar{\gamma}_{\rm II} = m_2$. Due to (A.6) *the equation* (A.7) *can be also written in the form*

$$\sum \mathsf{K}_{m_1}(\bar{\gamma}_{\rm I}|\bar{\alpha})\mathsf{K}_{m_2}(\bar{\beta}|\bar{\gamma}_{\rm II})\mathsf{f}(\bar{\gamma}_{\rm II}, \bar{\gamma}_{\rm I}) = (-q)^{m_2}\mathsf{f}(\bar{\beta}, \bar{\gamma})\mathsf{K}_{m_1+m_2}(\bar{\gamma}|\{\bar{\alpha}, \bar{\beta}q^2\}). \tag{A.8}$$

An analog of this lemma was proved in [24] (see appendix A of this work). The proof of (A.7) coincides with the one given in [24].

The equations (A.7), (A.8) yield similar identities involving $\mathsf{K}^{(l,r)}$, for instance,

$$\sum \mathsf{K}^{(l)}_{m_1}(\bar{\gamma}_{\rm I}|\bar{\alpha})\mathsf{K}^{(r)}_{m_2}(\bar{\beta}|\bar{\gamma}_{\rm II})\mathsf{f}(\bar{\gamma}_{\rm II}, \bar{\gamma}_{\rm I}) = (-q)^{m_2}\mathsf{f}(\bar{\beta}, \bar{\gamma})\mathsf{K}^{(l)}_{m_1+m_2}(\bar{\gamma}|\{\bar{\alpha}, \bar{\beta}q^2\})\,. \tag{A.9}$$

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